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The Nucleolus for Partially Defined Cooperative Games

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Abstract A partially defined cooperative game $PDCG$ consists of a set of players $N = \{1, 2, , n\}$ and a worth function w on a collection C of non-empty subsets of N, where $w(S)$ is the value that can be obtained from the cooperation of the coalition S. The Shapley value and the nucleolus are well-established fair allocation methods for when C equals all the non-empty subsets of N. There are fairness properties associated with each method. The nucleolus is rational, efficient, unbiased, subsidy-free, and consistent. Meanwhile, the Shapley value is marginal-monotone, subsidy-free, unbiased, additive, and efficient. The goal of this research was to find fair allocation methods for partially defined cooperative games when C consists only of coalitions of size $n, n-1, and$ 1. The group rationality, efficiency, and consistency of the obtained solution methods are studied.

Keywords allocation method · value · cooperative game · axioms · incomplete information

1 Introduction

2 Partially Defined Cooperative Games

Throughout this paper, we let $N = \{1, 2, ..., n\}$ be the fixed set of *players*. A nonempty subset S of N is called a *coalition*, and we write $|S|$ or s for the number of players in the coalition S. A *cooperative game* is a real-valued function $\hat{w}$ defined on the coalitions. By convention, we define $\hat{w}(\varnothing) = 0$. The real number $\hat{w}(S)$ is called the *worth* of coalition S and is interpreted as the total benefit available to the members of the coalition S if they cooperate with

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each other in the most efficient possible manner. In the context of a joint cost allocation problem, $\hat{w}(S)$ is the cost savings obtained through cooperation as opposed to each member working alone.

We often restrict the class of cooperative games under consideration. Collections of games often cited in the literature include zero-monotonic, superadditive, and convex. The cooperative game $\hat{w}$ is *zero-monotonic* if $\hat{w}(S) + \hat{w}(\{i\}) \leq \hat{w}(S \cup \{i\})$ for all coalitions S and players $i \in N - S$. The cooperative game $\hat{w}$ is *superadditive* if $\hat{w}(S) + \hat{w}(T) \leq \hat{w}(S \cup T)$ for all disjoint coalitions S and T. The cooperative game $\hat{w}$ is *convex* if $\hat{w}(S) + \hat{w}(T) \leq \hat{w}(S \cup T) + \hat{w}(S \cap T)$ for all coalitions S and T. Note that convex games are superadditive, and superadditive games are zero-monotonic. The cooperative game $\hat{w}$ is *zero-normalized* if $\hat{w}(\{i\}) = 0$ for all players $i \in N$.

Let $\hat{\Omega}$ be a collection of cooperative games. An *allocation method* on $\hat{\Omega}$ is a function $\varphi : \hat{\Omega} \to \mathbb{R}^n$. We interpret $\varphi_i(\hat{w})$ as the fair share to player i in the game $\hat{w}$. A well-known allocation method is the Shapley (1953) value defined by

$$\varphi_i(\hat{w}) = \sum_{S \subseteq N} \frac{(s-1)!(n-s)!}{n!} [\hat{w}(S) - \hat{w}(S - \{i\})] \tag{1}$$

In words, the Shapley value allocates to a player his average marginal contribution over all possible orderings of the players.

A partially defined cooperative game is a cooperative game in which only some of the coalition worths are known. In this paper, we will restrict our attention to games for which the worths of coalitions of sizes 1, $n-1$, and n are known, and the worths of singleton coalitions are always zero. Formally, a *partially defined cooperative game* (*pdcg*) is a vector of worths $w \in \mathbb{R}^{\{0\} \cup N}$. The real number w_0 is the worth of the grand coalition N, and w_i is the worth of the coalition $N - \{i\}$ for each player $i \in N$.

Since our viewpoint is that partially defined cooperative games arise when we have insufficient resources to determine the worth of each coalition, it is important to know what "fully defined" games could underlie a given partially defined game. Let $\hat{\Omega}$ be a collection of cooperative games. An *$\hat{\Omega}$-extension* of the pdcg w is a cooperative game $\hat{w} \in \hat{\Omega}$ satisfying $\hat{w}(\{N\}) = w_0$, $\hat{w}(N - \{i\}) = w_i$, and $\hat{w}(\{i\}) = 0$ for all $i \in N$. Define Ω to be the set of pdcg w that have an $\hat{\Omega}$-extension $\hat{w}$, and whatever word is used to describe a game in $\hat{\Omega}$ (e.g.,zero-monotonic) will also be used to describe a pdcg in Ω.

Example 1 Consider the four-player pdcg $w = (w_0, w_1, w_2, w_3, w_4)$. If $\hat{w}$ is an extension of w, then $\hat{w}(N) = w_0$, $\hat{w}(\{2,3,4\}) = w_1$, $\hat{w}(\{1,3,4\}) = w_2$, $\hat{w}(\{1,2,4\}) = w_3$, $\hat{w}(\{1,2,3\}) = w_4$, and $\hat{w}(\{1\}) = \hat{w}(\{2\}) = \hat{w}(\{3\}) = \hat{w}(\{4\}) = 0$. Furthermore, $\hat{w}$ is monotonic if and only if $0 \leq \hat{w}(\{i,j\}) \leq \min\{k,l\}$ for each of the three essentially different ways of assigning i,j,k,l to $1,2,3,4$ (e.g., $i=1$ and $j \in \{2,3,4\}$). Similarly, $\hat{w}$ is superadditive if and only if $\hat{w}$ is monotonic and $\hat{w}(\{i,j\}) + \hat{w}(\{k,l\}) \leq w_0$ for each of the three essentially different ways of assigning i,j,k,l to $1,2,3,4$. Finally, $\hat{w}$ is convex if and only if $\hat{w}$ is superadditive and $w_k + w_l - w_0 \leq \hat{w}(\{i,j\})$ for each of the six essentially different ways of assigning i,j,k,l to $1,2,3,4$.

More specifically, suppose $w = (9, 7, 6, 5, 4)$. An example of a convex (and so also monotonic and superadditive) extension has $\hat{w}(S) = 4$ for $|S| = 2$. An example of a monotonic and superadditive, but not convex, extension has $\hat{w}(S) = 0$ for $|S| = 2$. An example of a monotonic, but not superadditive or convex, extension has $\hat{w}(\{3,4\}) = 6$, $\hat{w}(\{1,2\}) = 4$, and $\hat{w}(\{2,4\}) = \hat{w}(\{1,4\}) = \hat{w}(\{2,3\}) = \hat{w}(\{1,3\}) = 0$.

Theorem 1 *The pdcg w is monotonic if and only if $0 \le w_i \le w_0$ for all $i \in N$.*

Proof If the pdcg w is monotonic, then there is a monotonic extension $\hat{w}$ of w, and so for any $i, j \in N$ with $i \ne j$, $0 = \hat{w}(\{j\}) \le \hat{w}(N - \{i\}) = w_i$, and $w_i = \hat{w}(N - \{i\}) \le \hat{w}(N) \le w_0$. Conversely, suppose $0 \le w_i \le w_{n+1}$ for all $i \in N$. Define the extension $\hat{w}$ by $\hat{w}(S) = 0$ if $|S| < n - 1$. Clearly, $\hat{w}$ is monotonic.

Theorem 2 *The pdcg w is superadditive if and only if $0 \le w_i \le w_0$ for all $i \in N$.*

Proof The proof is essentially identical to the proof of the previous theorem.

Theorem 3 *The pdcg w is convex if and only if $0 \le w_i \le w_0$ and $\sum_{j \in N - \{i\}} w_j \le (n-2)w_0$ for all $i \in N$.*

The following theorem was conjectured by Ramer. Unfortunately, her proof in the converse direction was flawed: the extension she defined is not always convex.

Proof If the pdcg w is convex, then there is a convex extension $\hat{w}$ of w. Then $\hat{w}(N - \{1\}) + \hat{w}(N - \{2\}) \le \hat{w}(N) + \hat{w}(N - \{1, 2\})$, $\hat{w}(N - \{1, 2\}) + \hat{w}(N - \{3\}) \le \hat{w}(N) + \hat{w}(N - \{1, 2, 3\})$, $\hat{w}(N - \{1, 2, 3\}) + \hat{w}(N - \{4\}) \le \hat{w}(N) + \hat{w}(N - \{1, 2, 3, 4\})$, ..., $\hat{w}(N - \{1, 2, \ldots, n-2\}) + \hat{w}(N - \{n-1\}) \le \hat{w}(N) + \hat{w}(N - \{1, 2, \ldots, n-1\}) = \hat{w}(N)$. Summing these $n - 2$ inequalities and simplifying, we obtain $\sum_{j \in N - \{n\}} \hat{w}(N - \{j\}) \le (n-2)\hat{w}(N)$, which is equivalent to $\sum_{j \in N - \{n\}} w_j \le (n-2)w_0$. Reordering the players so that some $i \in N$, instead of n, appears last, we obtain the condition in the theorem.

Conversely, suppose $0 \le w_i \le w_0$ and $\sum_{j \in N - \{i\}} w_j \le (n-2)w_0$ for all $i \in N$; these will be referred to as the *converse suppositions.* Define the extension $\hat{w}$ by $\hat{w}(S) = \max\{0, \sum_{i \in N - S} w_i - (n - s - 1)w_0$ if $1 < s < n - 1$. By this definition and the first converse supposition, $0 \le \hat{w}(S)$ for all coalitions S; this result will be referred to as *non-negativity.*

We claim that $\hat{w}(S) \le \hat{w}(T)$ for all coalitions $S \subseteq T$; this result will be referred to as *monotonicity.* Indeed, suppose $S \subseteq T$, and consider the following cases.

Case 1. Suppose $\hat{w}(S) = 0$. Then $\hat{w}(S) = 0 \le \hat{w}(T)$ follows by non-negativity.

Case 2. Suppose $\hat{w}(S) > 0$ and $|T| \le n - 1$. Then $\hat{w}(S) = \sum_{i \in N - S} w_i - (n-s-1)w_0$. By the first converse supposition, $0 \le -w_j + w_0$ for all $j \in T - S$.

Summing together the equality and inequalities from the last two sentences, we obtain $\hat{w}(S) \leq \sum_{i\in N-T} w_i - (n-t-1)w_0$. The right hand side of this inequality equals $\hat{w}(T)$: directly if $|T| = n-1$ and because it is positive if $|T| < n-1$. Thus, $\hat{w}(S) \leq \hat{w}(T)$.

Case 3. Suppose $\hat{w}(S) > 0$ and $T = N$. If $|S| = n-1$, then $S = N - \{i\}$ for some $i \in N$, and so $\hat{w}(S) = w_i \leq w_0 = \hat{w}(T)$ by the first converse supposition. If $|S| < n-1$, then $S \subseteq N-\{i\}$ for some $i \in N$. By Case 2, $\hat{w}(S) \leq \hat{w}(N-\{i\})$, and by the earlier result in this case, $\hat{w}(N-\{i\}) \leq \hat{w}(N)$. Thus, $\hat{w}(S) \leq \hat{w}(T)$.

We now claim the convexity condition: $\hat{w}(S)+\hat{w}(T) \leq \hat{w}(S\cup T)+\hat{w}(S\cap T)$ for all coalitions S and T. Indeed, suppose S and T are coalitions, and consider the following cases.

Case 1. Suppose $S \subseteq T$ (or $T \subseteq S$). Then $S \cup T = T$ and $S \cap T = S$ (or $S \cup T = S$ and $S \cap T = T$). Thus, $\hat{w}(S) + \hat{w}(T) = \hat{w}(S \cup T) + \hat{w}(S \cap T)$.

In the remaining cases, we will assume that neither coalition S or T is a subset of the other.

Case 2. Suppose $|S| = |T| = n-1$. Then $S = N - \{i\}$ and $T = N - \{j\}$ for some distinct $i, j \in N$. By its definition, $\hat{w}(N - \{i, j\}) \geq w_i + w_j - w_0$. Rearranging this inequality and using some definitions, we obtain $\hat{w}(S) + \hat{w}(T) = w_i + w_j \leq w_0 + \hat{w}(N - \{i, j\}) = \hat{w}(S \cup T) + \hat{w}(S \cap T)$.

Case 3. Suppose $|S| = n-1$ and $|T| < n-1$. Then $S = N - \{k\}$ for some $k \in N$, and since T is not a subset of S, it follows that $k \in T$. If $\hat{w}(T) = 0$, then $\hat{w}(S)+\hat{w}(T) = w_k+0 \leq w_0+0 = \hat{w}(N)+0 \leq \hat{w}(S\cup T)+\hat{w}(S\cap T)$. If $\hat{w}(T) > 0$, then $\hat{w}(S)+\hat{w}(T) = w_k+\sum_{i\in N-T} w_i-(n-t-1)w_0 = w_0+\sum_{i\in(N-T)\cup\{k\}} w_i-(n-t)w_0 = \hat{w}(N)+\sum_{i\in N-(S\cap T)} w_i-(n-|S\cap T|-1)w_0 \leq \hat{w}(S\cup T)+\hat{w}(S\cap T)$.

Case 4. Suppose $|T| = n-1$ and $|S| < n-1$. Follows immediately from Case 3 after interchanging S and T.

Case 5. Suppose $|S| < n-1$ and $|T| < n-1$. If $\hat{w}(S) = 0$, then $\hat{w}(S\cap T) = 0$ by non-negativity and monotonicity, and $\hat{w}(T) \leq \hat{w}(S \cup T)$ by monotonicity; hence, $\hat{w}(S) + \hat{w}(T) = 0 + \hat{w}(T) \leq \hat{w}(S \cap T) + \hat{w}(S \cup T)$. Similarly, the convexity inequality holds if $\hat{w}(T) = 0$. Now suppose $\hat{w}(S) > 0$ and $\hat{w}(T) > 0$. Then $\hat{w}(S)+\hat{w}(T) = \sum_{i\in N-S} w_i-(n-s-1)w_0+\sum_{i\in N-T} w_i-(n-t-1)w_0 = \left(\sum_{i\in N-(S\cup T)} w_i - (n-|S\cup T|-1)w_0\right)+\left(\sum_{i\in N-(S\cap T)} w_i - (n-|S\cap T|-1)w_0\right)$. In this last expression, the definition of $\hat{w}$ implies that the first term equals $\hat{w}(S\cup T)$ if $|S\cup T| \geq n-1$ and is no greater than $\hat{w}(S\cup T)$ if $|S\cup T| < n-1$, and the second term is no greater than $\hat{w}(S \cap T)$ if $|S\cap T| \geq 2$, equals $\sum_{i\in N-\{k\}} w_i - (n-2)w_0 \leq 0 = \hat{w}(S \cap T)$ for some $k \in N$ if $|S\cap T| = 1$ where the inequality follows from the second converse supposition, and equals $\sum_{i\in N} w_i - (n-1)w_0 \leq 0 = \hat{w}(S \cap T)$ if $|S\cap T| = 0$ where the inequality follows from both converse suppositions.

3 The Nucleolus for partially defined coalition games

For every allocation vector $x \in \Re^N$, and every coalition $S \subseteq N$, $e(x,S) = x(S) - v(S)$ is called the excess of coalition S at x.

Let (N, C, w) be a pdcg and let $K \subseteq \Re^N$ The nucleolus of the PDG (N, w) relative to K is the set $\aleph(N, w, K) = \{x \in K : \theta(x) \precsim_L \theta(y), \forall y \in K\}$ where $\theta(x)$ is the list of excesses $e(x, S) = \sum_{j \in S} x_j - w(S)$ for all $S \in C$ sorted in ascending order and $\precsim_L$ is lexicographic order. And, $\theta(x)$ is the excess vector.

Theorem 4 *Suppose $N = 1, 2, ..., n$ is a set of players, C contains the coalitions of size $n, n-1,$ and 1, and $(w(N) \geq w(N - \{n\}) \geq w(N - \{n-1\}) \geq \geq w(N - \{1\}) \geq w(\{n\}) = w(\{n-1\}) = ... = w(\{1\}) = 0$ are the known coalition worths. Then the nucleolus, x, for the partially defined cooperative game (N, C, w) can be computed in the following recursive manner.*

Case 1 If the game worths satisfy $w(N - \{i\}) \leq \left(\frac{n-2}{2}\right) w(N)$ then nucleolus is equal split i.e., $x_i = \frac{w(N)}{k}$, $\forall i \in N$ and $k = |N|$.

Case 2 If case 1 does not hold and if $w(N) \leq \frac{2\sum_{j \in N} w(N-\{j\}) - nw(N-\{i\})}{n-2}$, $\forall i \in N$, the nucleolus is $x_i = w(N) - w(N - \{i\}) + \lambda$ where $\lambda = \frac{\sum_{i \in N} w(n - \{i\}) - (n-1)w(N)}{n}$

Case 3 If case 2 does not hold, the last player gets $x_n = \frac{w(N) - w(N-n)}{2}$. The payoff for the remaining players is the nucleolus for the Davis-Maschler reduced game of coalition $\{1, 2, ..., n-1\}$.

Proof Suppose (N, C, w) is a zero-monotonic, partially defined cooperative game where $N = 1, 2, 3, .., n$. Let n be the weaker player. If the game satisfies the inequality in the first case, the allocation for player i is $x_i = \frac{w(N)}{k}$.

$$e(x, w(i)) = \frac{w(N)}{k} - 0 = \frac{w(N)}{k}$$

$$e(x, w(S)) = (n-1)\frac{w(N)}{k} - w(N - \{i\}) \leq \left(\frac{n-2}{2}\right) w(N)$$

Therefore, $e(x, w(i)) \leq e(x, w(S))$ where $S = N - \{i\}, i \in N$

So, if the game satisfies the first case, the allocation is the nucleolus.

If the game satisfy the inequality of the second case, the allocation method is $x_i = w(N) - w(N - \{i\}) + \lambda$ where $\lambda = \frac{\sum_{i \in N} w(n - \{i\}) - (n-1)w(N)}{n}$.

Since n is the weaker player, and $\frac{n-2}{n} w(N) < w(N - \{i\}) \leq \frac{2\sum_{j \in N} w(N-\{j\}) - nw(n-\{i\})}{n-2}$

Then, $e(x, w(S)) \leq e(x, w(i)), S \subset N$

If the game satisfy the third case, then the nth player will get half of their marginal, and since we assumed that the nth player is the weakest player, then their excess will be the least among other players.

Then we will redefine the nucleolus using the Davis- Maschler reduced game property.

When we reapply the theorem to the new game, there are five possible cases.

Case 4 The remaining player will each get $x_i = \dfrac{w(N) - x_n}{k-1}$. Since n is the weakest player and all the remaining players get the same amount because the reduced game satisfy $w(N - \{i\}) \leq \left(\dfrac{n-2}{2}\right) w(N)$, the following inequality holds.

$$e(x_n, w(n)) \leq e(x_i, w(i)) \leq e(x_{N-\{i\}}, w(N-\{i\})) \leq e(x_{N-\{n\}}, w(N-\{n\})), i \neq n$$

Therefore, it is the nucleolus.

Additionally, the excesses of the reduced game will be the same as the excess of the original game since we subtract the payoff of player n from the grand coalition and the coalitions that includes the player.

The excess inequality will also hold if more than one player get half of their marginal.

Case 5 The remaining player will each get $x_i = w(N) - w(N-i) + \lambda$ where $\lambda = \dfrac{\sum_{i\in N} w(n-i) - (n-1)w(N)}{n}$ n is the weaker player and every $N - \{i\} - \{n\}$ coalition satisfy the following inequality.

$$\frac{n-2}{n} w(N) < w(N - \{i\}) \leq \frac{2\sum_{j\in N w(N-\{j\})-nw(n-\{i\})}}{n-2}$$

Therefore, the following excesses inequality holds.

$$e(x, w(S)) \leq e(x, w(i)), S \subset N$$

The previous inequality will also hold for the case where more than one player gets half of their marginal.

Example 2 Let (N, C, w) be a partially defined cooperative game where $N = 1, 2, 3, 4$.

$$w(1) = w(2) = w(3) = w(4) = 0$$

$$w(N) = 100, w(123) = 50, w(134) = 40, w(124) = 30, w(234) = 20$$

$$w(N-i) \leq \frac{w(N)}{2}, \forall i \in N$$

The game satisfies the inequality of the first case. Therefore, the nucleolus can be computed in the following way.

$$x_i = \frac{w(N)}{4} = \frac{100}{4} = 25$$

$e(x_{123}, w(123)) = 25, e(x_{124}, w(124)) = 45, e(x_{134}, w(134)) = 35, e(x_{234}, w(234)) = 55, e(x_1, w(1)) = 25, e(x_2, w(2)) = 25, e(x_3, w(3)) = 25, e(x_4, w(4)) = 25$

Example 3 Let (N, C, w) be a partially defined cooperative game where $N = 1, 2, 3, 4$.

$$w(1) = w(2) = w(3) = w(4) = 0$$

$$w(N) = 100, w(123) = 75, w(134) = 70, w(124) = 60, w(234) = 55$$

$$w(N-i) \leq \sum_{j \in N} w(N-j) - 2w(N-i), \forall i \in N$$

The game satisfies the inequality of the second case. Therefore, the nucleolus can be computed in the following way.

$$x_1 = 35, x_2 = 30, x_3, = 20, x_4 = 15$$

$e(x_{123}, w(123)) = 10, e(x_{124}, w(124)) = 10, e(x_{134}, w(134)) = 10, e(x_{234}, w(234)) = 10, e(x_1, w(1)) = 35, e(x_2, w(2)) = 30, e(x_3, w(3)) = 20, e(x_4, w(4)) = 15$

Example 4 Let (N, C, w) be a partially defined cooperative game where $N = 1, 2, 3, 4$.

$$w(1) = w(2) = w(3) = w(4) = 0$$

$$w(N) = 1000, w(123) = 800, w(134) = 160, w(124) = 540, w(234) = 100$$

The game doe not satisfy the inequalities of any of the two games, so player 4 gets $x_4 = \frac{1000 - 800}{2} = 100$.

The reduced game can be defined in the following way.

$$w(1) = w(2) = w(3) = w(4) = 0$$

$$w'(123) = 900, w'(12) = 440, w'(13) = 60, w'(23) = w'(1) = w'(2) = w'(3) = 0$$

The reduced game does not satisfy the first two cases as well, so the last player gets $x_3 = \frac{900 - 440}{2} = 230$ And, the new reduced game can be defined in the following mannar.

$$w"(12) = 670, w"(1) = 0, w"(2) = 0$$

The definition of the nucleolus on a fully-defined, two-player games yields the payoff for the remaining two players.

$$x_1 = x_2 = \frac{w"(12) + w"(1) - w"(2)}{2} = \frac{w"(12) - w"(1) + w"(2)}{2} = 335$$

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