A Coalition Game on Finite Groups

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1 Introduction

Group theory is a powerful tool for understanding and characterizing symmetries. Coalition games are useful to model real-life situations where a fair allocation of resources or savings is needed. It is not immediately clear that there should be a direct relation between groups and coalition games. However, when a coalition game is defined on a group, certain patterns are revealed in relation to the group. Previous research on the subject was not found.

In this paper we examine a conjecture about the prenucleolus of coalition games defined on the finite group $\mathcal{Z}_n$ (integers mod n with addition) through certain examples.

In section 2, we define a group. We define coalition games in section 3. Lastely, section 4 has the conjecture and the examples.

2 Introduction to Groups

The following are important definitions and theorems that are utilized through the examples presented.

We start by defining a group and a group generator.

Definition 1. A *group* is a set G together with a binary operation $(\cdot)$ such that the following four properties hold:

1. (closure) For any x and y in G, $x \cdot y$ is in G.

2. (identity) There exists a member e in G which has the property that, for all x in G, $e \cdot x = x \cdot e = x$.

3. (inverse) For every x in G, there exists a y in G, called the inverse of x, such that $x \cdot y = e$.

4. (associative law) For any x, y, and z in G, then $(x \cdot y) \cdot z = x \cdot (y \cdot z)$.

Definition 2. An element g of the group G is said to be a generator if every element of G can be expressed as a power of g.

The following definitions provide a way we can find the generators of any $\mathcal{Z}_n$ groups. They also provide a method of obtaining the size of the set of generators any $\mathcal{Z}_n$ groups.

Theorem 3. For the group of integers mod n with addition as the binary operation $\mathcal{Z}_n$, the generators of $\mathcal{Z}_n$ are precisely the integers between 0 and n that are co-prime to n.

The following theorem gives us a way to find the size of a set of generators of any $\mathcal{Z}_n$ group.

Theorem 4. If the prime factorization of n is given by

$$n = p_1^{r_1} \cdot p_2^{r_2} \dots p_k^{r_k},$$

where $p_1, p_2, \dots, p_k$ are distinct primes, and $r_1, r_2, \dots, r_k$ are positive integers, then the count of numbers less than n that are coprime to n is

$$\phi(n) = (p_1 - 1) \cdot p_1^{(r_1 - 1)} \cdot (p_2 - 1) \cdot p_2^{(r_2 - 1)} \dots \ (p_k - 1) \cdot p_k^{(r_k - 1)}.$$

The following are some important theorems that will be utilized in the examples provided.

Theorem 5. Let G be a finite group, and H a subgroup of G. Then the order of H divides the order of G. That is, $|G| = k \cdot |H|$ for some positive integer k.

Theorem 6. Given two non-zero integers x and y, the greatest common divisor of x and y is the smallest positive integer that can be expressed in the form $ux + vy$ with u and v being integers.

Since we covered the needed definitions and theorems from group theory, we will define what are coalition games and how we solve them in the following section.

3 Coalition Games

Definition 7. A *coalition game* consists of a finite set N of n players (where $n = |N|$) and a real-value function w on the non-empty subsets of N. A non-empty subset of N is called a *coalition*, and $w(S)$ is the *worth* of coalition S.

You can think of the players of a coalition game as companies, cities, or individuals who have to accomplish a certain task. We can think about the savings each subset of our players (or coalitions of players) can have by working together. The amount of savings is represented by the worth of each coalition. Solving a coalition game means finding an allocation. An allocation can be defined in the following way.

Definition 8. An *allocation* for the coalition game N with a worth function w is a payoff vector $x = (x_1, x_2, \dots, x_n)$ satisfying $\sum_{i \in N} x_i = w(N)$.

Allocations can be thought of as a fair way to divide the savings players make when they all work together. It can be seen as a way to encourage players to join the Grand Coalition. One way we judge an allocation is through the excess each player has. Excess can be defined in the following way.

Definition 9. For every allocation x, and every coalition $S \subseteq N$,

$$e(S, x) = x(S) - w(S)$$

is called the *excess* of coalition S at x.

An excess can be thought of as a measure of a coalition happiness. The bigger the coalition excess is, the more the coalition is getting above their worth, so the happier the coalition must be. One of the allocations that try to make sure tht the coalitional excess is as big as possible for all coalitions is the prenucleolus. It is defined below.

Definition 10. The *prenucleolus* ν is the allocation that successively maximizes the smallest excesses. More formally, for a given coalition game (N, w) and payoff $x = \{x_1, \dots, x_n\}$, we define the 2^n-vector $\theta(x)$ as the vector whose components are the excesses of the coalitions $S \subset N$ sorted in a non decreasing order. The nucleolus ν lexicographically maximizes $\theta(x)$ for all x.

Now that we covered some important definitions about games, we can move on to defining a coalition game on finite groups.

4 Group Coalition Games

Definition 11. A *coalition game on the finite group* (G, w) consists of players which are represented by the elements of the group G and a worth function w defined on the nonempty subsets S of G.

Theorem 12. The prenucleolus payoff ν to every element i of the group G is $\nu_i \geq 0$

Proof. Let $i \in G$ where G is a group of order n. Consider the coalition $S \subset G$. It follows that $w(S) \geq w(S - \{i\})$.

Suppose $\nu_i < 0$, the excess

$$\begin{aligned} e(\nu, S) = \sum_{i \in S} \nu_i - w(S) &\leq \sum_{j \in S - \{i\}} \nu_i + \nu_j - w(S - \{i\}) \\ &= e(\nu, S - \{i\}) + \nu_i < e(\nu, S - \{i\}) \end{aligned}$$

.

Hence, the minimum excess is for one or more coalitions containing i, and the excesses for coalitions that do not contain i are strictly larger than the minimum excess. Hence, taking away a very small amount from each player $j \neq i$ and giving it all to player i will raise the minimum excess. So, x could not have been the prenucleolus. Since the only extra assumption we made for the allocation ν was $\nu_i < 0$, it follows that $\nu_i(w) \geq 0$. □

The following is a conjecture defining the prenucleolus for $\mathcal{Z}_n$ groups.

Conjecture 1. For a group G, the set A is the set of generators, set C is the largest proper subgroup of G generated by the smallest prime factor p_1 of G, and set B has all the elements not in A or C.

The prenucleolus ν for $i \in A$, $j \in B$ and $k \in C$, is

$$\begin{cases} \nu_i = \nu_j = \dfrac{p_1}{p_1 - 1}, \nu_k = 0 & \text{if } |B| < |C| \\ \\ \nu_i = \lambda, \nu_j = \nu_k = \dfrac{\lambda}{2} \ \forall k \neq \{e\}. & \text{if } |B| > |C| \end{cases}$$

And, $|B|$ cannot equal $|C|$.

Now that we reviewed the conjecture, we can look at some examples and see if the conjectured prenucleolus is the prenucleolus for these examples.

Example 1. Consider the group $\mathcal{Z}_{105}$, the coalition game is defined in the following way.

$$w(S) = \begin{cases} 105 & \text{if } (S \cap A \neq \emptyset) \text{ or two elements are coprime} \\ 35 & \text{if } S \subset \{3, 6, 9, 12, 18, 24, 27, 33, 36, 39, 48, 51, 54, 57, 66, 69, 72, 78, 81, 87, 93, 96, 99, 102\} \\ 21 & \text{if } S \subset \{5, 10, 20, 25, 40, 50, 55, 65, 80, 85, 95, 100\} \\ 15 & \text{if } S \subset \{7, 14, 28, 49, 56, 77, 91, 98\} \\ 7 & \text{if } S \subset \{15, 30, 45, 60, 75, 90\} \\ 5 & \text{if } S \subset \{21, 42, 63, 84\} \\ 3 & \text{if } S \subset \{35, 70\} \\ 1 & \text{if } S = \{0\} \end{cases}$$

Observe that $|A| = \phi(105) = 48$, $|C| = [3] = 35$, so $|B| = 22$. According to the conjectured prenucleolus, $\nu_i = \nu_j = 3/2$ and $\nu_k = 0$.

By the definition of the coalition game and Theorem 6, the worth function w satisfies $w(S) = 105$ if S generates $\mathbb{Z}_{105}$ and $1 \leq w(S) \leq 35$ otherwise.

Notice $\displaystyle\sum_{i=0}^{104} \nu_i = 48(3/2) + 22(3/2) + 35(0) = 105$. Hence, ν is an allocation.

Let C_0 be the elements of C divisible by neither 7 nor 5, $|C_0| = 24$ which is larger than $|B|$.

Observe that the minimal generating sets include
- the singletons from A, and
- pairs consisting of one element from B and one element from C_0.

Indeed, it is well known that the an element i of $\mathbb{Z}_n$ is a generator if and only if i is coprime to n (Theorem 3). This verifies the first collection. It is also well known that if a and b are non-zero integers, there exist integers u and v satisfying $ua + vb = \gcd(a, b)$, the greatest common divisor of a and b (Theorem 6). Since i and j are coprime in the second collection, i and j generate 1 which then generates the entire group. This verifies the second collection.

Denote the union of the two described collections by $\mathcal{M}$.

Furthermore, observe that any subset of C does not generate the group (e.g., 1 is not a multiple of 3 and so cannot be generated by C). Thus, any coalition $S \notin \mathcal{M}$ must either not generate the group or must contain at least one element of $A \cup B$.

Now observe that the excess of the conjectured prenucleolus for each coalition $S \in \mathcal{M}$ is $\chi(\nu, S) = 1.5 - 105 = -103.5$. If $S \subset C$, then $\chi(\nu, S) = 0 - w(S) \geq -35$. Finally, if $S \notin \mathcal{M}$ is not a subset of C, then $\chi(\nu, S) \geq 1.5 - 105 = -103.5$. Thus, the minimum excess for the allocation ν is -103.5.

Now suppose that x is the prenucleolus. By the previous paragraph, the minimum excess of x must be at least -103.5. Specifically,
- $\chi(x, \{i\}) \geq -103.5$ if $i \in A$; and
- $\chi(x, \{i, j\}) \geq -103.5$ if $i \in B$ and $j \in C_0$;.

By plugging in the definition of excess and the worth function and simplifying, we obtain
- $x_i \geq 1.5$ if $i \in A$;
- $x_i + x_j \geq 1.5$ if $i \in B$ and $j \in C_0$.

By a Theorem 14, the elements of x must be nonnegative. Specifically, - $x_i \geq 0$ if $i \in C$.

Finally, since x is an allocation, we have - $\sum_{i=1}^{104} x_i = 105$.

We organize the inequalities in the first column of the following table. The next eight columns give the number of inequalities that contain x_i for each i in the set specified in the header row. The 'number' column gives the number of inequalities of each type.

Inequalities		A	B	C_0	$C - C_0$	number	weight
$x_i \geq 1.5,$	$i \in A$	1	0	0	0	48	1
$x_i + x_j \geq 1.5,$	$i \in B, j \in C_0$	0	24	22	0	$22 \cdot 24 = 528$	1/24
$x_i \geq 0,$	$i \in C_0$	0	0	1	0	24	1/12
$x_i \geq 0,$	$i \in C - C_5$	0	0	0	1	11	1

Multipling each inequality of each type by the corresponding weight and summing, we obtain $\sum_{i=1}^{104} x_i \geq 105$. Now the left hand side equals 105, and so all of the inequalities must hold with equality. This implies first that $x_i = 0$ for all $i \in C$, and then this implies that $x_i = 1.5$ for all $i \in A \cup B$. Thus, $x = \nu$.

The case in this example had a stronger condition than the conjecture: that is $|B| < |C_0|$. Later, the result in this example will be generalized to all group coalition games where $|B| < |C_0|$.

Example 2. Consider the group $\mathcal{Z}_{2431}$, the coalition game is defined in the following way.

$$w(S) = \begin{cases} 2431 & \text{if } (S \cap A \neq \emptyset) \text{ or two elements are coprime} \\ 221 & \text{if } S \subset \{11, 22, 33, 44, \ldots, 2420\} \\ 187 & \text{if } S \subset \{13, 26, 39, 52, \ldots, 2418\} \\ 143 & \text{if } S \subset \{17, 34, 51, 68, 85, \ldots, 2414\} \\ 17 & \text{if } S \subset \{143, 286, 429, 572, 715, \ldots, 2288\} \\ 13 & \text{if } S \subset \{187, 374, 561, 748, \ldots, 2244\} \\ 11 & \text{if } S \subset \{221, 442, 663, \ldots, 2210\} \\ 1 & \text{if } S = \{0\} \end{cases}$$

Observe that $|A| = \phi(2431) = 1920$, $|C| = [11] = 221$, so $|B| = 290$. According to the conjectured prenucleolus, $\nu_i = \lambda = \dfrac{4862}{4351}$ and $\nu_j = \nu_k = \dfrac{\lambda}{2} = \dfrac{4862}{8702}$.

By the definition of the coalition game and Theorem 4, the worth function w satisfies $w(S) = 2431$ if S generates $\mathbb{Z}_{105}$ and $1 \leq w(S) \leq 221$ otherwise.

Notice $\displaystyle\sum_{i=0}^{2430} \nu_i = 1920 \left(\frac{4862}{4351}\right) + 290 \left(\frac{4862}{8702}\right) + 221 \left(\frac{4862}{8702}\right) = 2431$. Hence, ν is an allocation.

Let us consider subsets of B and C to identify the minimal generating sets of G.
Set B can be divided in
- Set nny
where the elements of the set are only divisible by the third prime factor of $|G|$: 17 (observe the y in the third place).

$$|nny| = (p_1 - 1) \cdot p_1^{(r_1-1)} \cdot (p_2 - 1) \cdot p_2^{(r_2-1)} = 10 * 12 = 120$$

- Set nyn
where the elements of the set are only divisible by the second prime factor of $|G|$: 13 (observe the y in the second place).

$$|nyn| = (p_1 - 1) \cdot p_1^{(r_1-1)} \cdot (p_3 - 1) \cdot p_3^{(r_3-1)} = 10 * 16 = 160$$

- Set nyy
where the elements of the set are only divisible by the product of the second and the third prime factor of $|G|$: 221 (observe the y in the second and third place).

$$|nyy| = (p_1 - 1) \cdot p_1^{(r_1-1)} = 10$$

The orders of these subsets add up to the order of $|B|$ and each is divisable by a distinct number, then the three subsets partition B.

Set $C - \{e\}$ can be divided in
- Set ynn
where the elements of the set are only divisible by the first prime factor of $|G|$: 1 (observe the y in the first place).

$$|ynn| = (p_2 - 1) \cdot p_2^{(r_2-1)} \cdot (p_3 - 1) \cdot p_3^{(r_3-1)} = 12 * 16 = 192$$

- Set yyn
where the elements of the set are only divisible by the product of the first and the second prime factor of $|G|$: 143 (observe the y in the first and second place).

$$|yyn| = (p_3 - 1) \cdot p_3^{(r_3-1)} = 16$$

- Set yny
where the elements of the set are only divisible by the product of the first and the third prime factor of $|G|$: 170 (observe the y in the first and third place).

$$|yny| = (p_2 - 1) \cdot p_2^{(r_2-1)} = 12$$

The orders of these subsets add up to the order of $|C| - 1$ and each is divisable by a distinct number, then the three subsets partition $C - \{e\}$.

Observe that the minimal generating sets include
- the singletons from set A,
- pairs consisting of one element from set nny and one element from set ynn,
-pairs consisting of one element from set nny and one element from set yyn,
- pairs consisting of one element from set nyn and one element from set ynn,
- pairs consisting of one element from set nyn and one element from set yny,
-pairs consisting of one element from set nyy and one element from set ynn,
-pairs consisting of one element from set nny and one element from set nyn.

The first collection can be verified using Theorem 4. The remaining collections can be verified using Theorem 7 since they all consist of two elements that are coprime, so the pairs generate the group.

Denote the union of these collections by $\mathcal{M}$

Furthermore, observe that any subset of C does not generate the group (e.g., 1 is not a multiple of 11 and so cannot be generated by C). Thus, any coalition $S \notin \mathcal{M}$ must either not generate the group or must contain at least one element of $A \cup B$.

Now observe that the excess of the conjectured prenucleolus for each coalition $S \in \mathcal{M}$ is $\chi(\nu, S) = \frac{4862}{4351} - 2431 = -\frac{10,572,419}{4351}$. If $S \subset C$, then $\chi(\nu, S) = 0 - w(S) \geq -221$. Finally, if $S \notin \mathcal{M}$ is not a subset of C, then $\chi(\nu, S) \geq \frac{4862}{4351} - 2431 = -\frac{10,572,419}{4351}$. Thus, the minimum excess for the allocation ν is $-\frac{10,572,419}{4351}$.

Now suppose that y is the prenucleolus. By the previous paragraph, the minimum excess of y must be at least $-\frac{10,572,419}{4351}$. Specifically, - $\chi(y, \{i\}) \geq -\frac{10,572,419}{4351}$ if $i \in A$; and

- $\chi(y, \{i, j\}) \geq -\frac{10,572,419}{4351}$ if $i \in nny$ and $j \in ynn$; and
- $\chi(y, \{i, j\}) \geq -\frac{10,572,419}{4351}$ if $i \in nny$ and $j \in yyn$; and
- $\chi(y, \{i, j\}) \geq -\frac{10,572,419}{4351}$ if $i \in nyn$ and $j \in ynn$; and
- $\chi(y, \{i, j\}) \geq -\frac{10,572,419}{4351}$ if $i \in nyn$ and $j \in yny$; and
- $\chi(y, \{i, j\}) \geq -\frac{10,572,419}{4351}$ if $i \in nyy$ and $j \in ynn$; and
- $\chi(y, \{i, j\}) \geq -\frac{10,572,419}{4351}$ if $i \in nny$ and $j \in nyn$.

Substituting in the definition of excess and the worth function and simplifying, we get

- $y_i \geq \frac{4862}{4351}$ if $i \in A$
- $y_i + y_j \geq \frac{4862}{4351}$ if $i \in nny$ and $j \in ynn$
- $y_i + y_j \geq \frac{4862}{4351}$ if $i \in nny$ and $j \in yyn$
- $y_i + y_j \geq \frac{4862}{4351}$ if $i \in nyn$ and $j \in ynn$

- $y_i + y_j \geq \frac{4862}{4351}$ if $i \in nyn$ and $j \in yny$
- $y_i + y_j \geq \frac{4862}{4351}$ if $i \in nyy$ and $j \in ynn$
- $y_i + y_j \geq \frac{4862}{4351}$i $\in nny$ and $j \in nyn$

We organize the inequalities in the first column of the following table. The next seven columns give the number of inequalities that contain y_i for each i in the set specified in the header row. The 'number' column gives the number of inequalities of each type.

Inequalities		A	B	C	number	weight
$y_i \geq \frac{4862}{4351}$,	$i \in A$	1	0	0	1920	1
$y_i + y_j \geq \frac{4862}{4351}$,	$i \in nny, j \in ynn$	0	120	192	$120 \cdot 192 = 23040$	α
$y_i + y_j \geq \frac{4862}{4351}$,	$i \in nny, j \in yyn$	0	120	16	$120 \cdot 16 = 1920$	1/120
$y_i + y_j \geq \frac{4862}{4351}$,	$i \in nyn, j \in ynn$	0	160	192	$160 \cdot 192 = 30720$	β
$y_i + y_j \geq \frac{4862}{4351}$,	$i \in nyn, j \in yny$	0	160	12	$160 \cdot 12 = 1920$	1/160
$y_i + y_j \geq \frac{4862}{4351}$,	$i \in nyy, j \in ynn$	0	10	192	$10 \cdot 192 = 1920$	1/192
$y_i + y_j \geq \frac{4862}{4351}$,	$i \in nny, j \in nyn$	0	280	0	$120 \cdot 160 = 19200$	γ

We need to find coefficients α, β, and γ such that when we multiply each inequality by its weight and sum over all inequalities, every element would be counted only once.

Therefore, the needed coefficients can be obtained through solving the following equations.

$$160\alpha + 192\beta + \frac{16}{120} = 1$$

$$160\gamma + 120\beta + \frac{20}{192} = 1$$

$$120\alpha + 192\gamma + \frac{12}{160} = 1$$

Therefore, $\alpha = \frac{1}{480}$, $\beta = \frac{1}{360}$, and $\gamma = \frac{9}{2560}$.

Multiplying every inequality by its weight and summing over, we obtain $\sum_{i=1}^{104} x_i \geq 2431$. The left hand side equals 2431, so all inequalities must hold with equality. Then, $y_i = \frac{4862}{4351}$ for $i \in A$ and $y_j = \frac{4862}{8702}$ for $j \in B \cup C$. Thus $y = x$.

The examples provided are indeed not sufficient to prove the conjecture. However, for the sake of keeping this paper as short as possible, proofs ofparts of the conjecture are not included.