Quota Violations in the Apportionment Method of Webster

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1 Introduction

Apportionment is the process by which seats in a legislative body are distributed among administrative divisions entitled to representation. The United states uses apportionment to distribute seats from the House of Representatives to the states. The goal is to apportion seats as fairly as possible, meaning that the ideal is the ratio of 1 person: 1 vote. There exist different methods to decide how to apportion seats. This paper uses Webster's method to explore how often a state gets an "unfair" apportionment of seats and the conditions when this occurs.

2 Research framework

An *apportionment problem* consists of a positive integer number of states s, a vector of positive real populations $\mathbf{p} = (p_1, p_2, \ldots, p_s)$, a nonnegative integer house size h, and a vector of nonnegative integer minimum requirements $\mathbf{r} = (r_1, r_2, \ldots, r_s)$. For example, the US house of represenatives has a minimum requirement of 1 and a house size of 435.

An *apportionment* for the described apportionment problem is a vector of integers $\mathbf{a} = (a_1, a_2, \ldots, a_s)$ such that $a_1 + a_2 + \cdots + a_s = h$ and $a_i \geq r_i$ for $i = 1, 2, \ldots, s$.

Other parameters used in this paper are a divisor, λ that changes due to the method. A quotient vector, $\mathbf{x} = (x_1, x_2, \ldots, x_s)$ defined by $x_i = p_i/\lambda_0$ for $i = 1, 2, \ldots, s$. A quota vector $\mathbf{q} = (q_1, q_2, \ldots, q_s)$, defined by $\mathbf{q} = (p_i \div \sum_{i=1}^{s} p_i)h$ for $i = 1, 2, \ldots, s$. Notice that the quotient vector can change with the a change of the divisor λ whereas the quota vector is found with the population vector and house size which do not change. I will also define a function $\rho(x) = x - \lfloor x \rfloor$ that allows us to reference the fraction of a number.

3 Methods

An *apportionment method* is a function from apportionment problems to sets of apportionments. For example Hamiltons method:

1. Find the quotas and give to each state the whole number contained in its quota.

2. Assign any seats which are as yet apportioned to those states having the largest fractions or remainders.

One possibility for Hamilton in step 2: If two or more state fractions are equal, then the extra seat can be assigned to any one of these states (e.g., by rolling a fair die).

Hamilton's method apportions based directly on state quotas, divisor methods vary a divisor, , to obtain a quotient for each state, the *quotients* are rounded up or down in a specified manner, and the divisor is chosen so that the rounded quotients sum to the house size.

Webster's method is:

1. Find a divisor so that the whole numbers nearest to the quotients of the states sum to the house size.

2. Give to each state its whole number.

One possibility for Webster in step 1: If two or more fractions of state quotients are equal to 1/2, then each can be rounded up or down in the attempt to obtain the desired sum.

An apportionment *breaks quota* if at least one state receives more than their quota rounded up or less than their quota rounded down. In this paper we will explore Webster's mehtod and when it breaks quota.

4 Theorem. If there are two states, then the apportionment vectors obtained by Webster's method never break quota.

Proof.

First, s = 2 and let λ_0 be the initial divisor found by $\lambda_{\mathbf{0}} = \sum_{i=1}^{s} p_i \div h$. This means by using the initial divisor, the quotients and quotas will be equal.

$$x_i = \frac{p_i}{\lambda_0} = \frac{\sum_{i=1}^{s} p_i}{\frac{\sum_{i=1}^{s} p_i}{h}} = \frac{p_i}{\sum_{i=1}^{s} p_i} * h = q_i$$

for $i = 1, 2$. The quotient vector will sum to the size of the house and the fractions will either sum to 0 or 1.

Case 1: If the fractions sum to 0 then both numbers are integers and therefore is within quota.

Case 2: If the fractions sum to 1 and do not equal .5 then we round appropriately and the vector will sum to the total house size and therefore is within quota.

Case 3: If the fractions are both .5. Then there are two solutions that satisfy quota and a fair method will be used to determine the answer.

Thus Webster's method does not break quota when there are two states.

5 Theorem. If there are three states, then the apportionment vectors obtained by Webster's method never break quota.

Proof.

First, s = 3 and let λ_0 be the initial divisor found by $\lambda_{\mathbf{0}} = \sum_{i=1}^{s} p_i \div h$. This means by using the initial divisor, the quotients and quotas will be equal.

$$x_i = \frac{p_i}{\lambda_0} = \frac{\sum_{i=1}^{s} p_i}{\frac{\sum_{i=1}^{s} p_i}{h}} = \frac{p_i}{\sum_{i=1}^{s} p_i} * h = q_i$$

for $i = 1, 2, 3$. The quotient vector will sum to the size of the house and the fractions will either sum to 0, 1 or 2.

Case 1: If the fractions sum to 0 then both numbers are integers (or whole numbers) and therefore are within quota.

Case 2: If the fractions sum to 1 then the whole numbers sum to $(h - 1)$, so to satisfy quota one state must roundup while the other two round down.

If one fraction is .5 or greater and the other fractions are less than .5 then one rounds up and the others round down and therefore satisfying quota.

If all the fractions are less than .5 then we must change the divisor, λ. So, decrease the divisor to increase the quotients until the fraction vector fits the conditions of the first example.

If there are two fractions of .5 then there are two possible solutions that satisfy quota and a fair method must be defined to determine the solution.

Case 3: If the fractions sum to 2 then the whole numbers sum to $(h - 2)$, so to satisfy quota two states must roundup while the other rounds down.

If one fraction is .5 or greater and the other is greater than .5 they round up and the other must be less than .5 and rounds down therefore satisfying quota.

If all the fractions are .5 or greater then we will need to increase the divisor to decrease the quotients fraction vector until it fits the conditions of the first example.

Thus Webster's method will satisfy quota when there are three states.

6 Example

Now we will explore an example involving four states. In this example the house size is 35. The table below shows the four states, their populations, quota, Hamilton's apportionment, quotient and Webster's apportionment. The quotient was found by choosing an initial divisor $\lambda_{\mathbf{0}} = \sum_{i=1}^{s} p_i \div h = 45527.3$, meaning the quotas and quotients will initially be equal. When we round then sum the initial quotients the total is 36, one more than the size of the house. Therefore I increased the divisor to decrease the quotients until the rounded sum of the quotients is the house size of 35.

State	Population	Quota	Hamilton	Quotient	Webster
A	70,653	1.552	1	1.507	2
B	117,404	2.579	3	2.505	3
C	210,923	4.633	5	4.501	5
D	1,194,456	26.236	26	24.489	25
Total	1,593,436	35	35	35	35

Notice in the table above table state D has a quota of 26.236 but Webster's method only apportions 25 seats to state D. Thus Webster's method will not always satisfy quota when there are four states.

7 Conjecture for four states

By running simulation experiments that generate a quota vector uniformly from a simplex and a hypercube, I was able to conclude that Webster's method violates quota when there are many small states and one large state. Webster's method always satisfies quota in a two state example and three state example so I fixed the number of states at four and increased the house size from six to ten. The simulation returns a list of possible apportionment vectors, a list of the number of quota vectors with the corresponding apportionment vectors, and a list of the number of quota violations for the corresponding apportionment vector. The apportionment vector's returned that violated quota from the simplex and hypercube were always cases of having three small states and one large state. For example, when s = 4 and h = 6, the simplex simulation returns (0, 0, 0, 6), (0, 0, 1, 5), (1, 1, 1, 3) as the apportionment vectors that violate quota. Both simulations generated 1,000,000 quota vectors.

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