A Value for Zero-monotonic Partially Defined Games

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Introduction

A *cooperative game* is a pair (N,v) where N={1,2,...,n} and v is a real-valued function on the nonempty subsets of N. The elements of N are generally called *players*, and the subsets of N are called *coalitions*. We call v the *worth function*, and so v(S) is interpreted as the worth of the coalition S. In other words, v(S) is the amount that the players in S can jointly produce through cooperation.

The *zero-normalization* of a game (N,v) is the game (N,u) where $u(S) = v(S) - \sum_{i\in S} v(\{i\})$. It follows that a game is zero-normalized if v({i})=0; that is, the worth of each singleton coalition is equal to zero. A game (N,v) is *monotonic* if $v(S) \leq v(T)$ for all $S\subseteq T\subseteq N$. Moreover, a game is said to be *zero-monotonic* if its zero-normalization is monotonic. Monotonicity is a favorable property as it assures that the addition of a player will not lessen the worth of a coalition. We shall assume zero-normalization and zero-monotonicity throughout this report.

One goal of cooperative game theory is to find a fair method of distributing joint savings or costs among the players involved in a venture. An *allocation method*, or *value,* is a function that assigns to each game (N,v) an *allocation* $x = (x_1,x_2,\ldots,x_n)$ where x_i is the fair share, or *payoff*, of the total benefit v(N) that player i receives for its cooperation in the group. One commonly used allocation method is the Shapley value, which is given by the formula:

$$\phi_i(N,v) = \sum_{S \ni i, S \subseteq N} \frac{(s-1)!\,(n-s)!}{n!}[v(S) - v(S - \{i\})]$$

where $s=|S|$ and $n=|N|$. The Shapley value, which assigns to each player the average of its marginal contribution over all possible orderings of players, is a linear transformation from the space of games in $\Re^{2^n-1}$ to the space of allocations in $\Re^n$.

Partially defined games

For every cooperative game there are 2^n-1 possible coalitions. As this number increases exponentially with n, it can become impractical or expensive to determine the worths of all coalitions when the number of players is large. We are therefore interested in determining a method for computing payoffs when only limited information is known.

Cooperative games in which some of the coalitional worths are unknown are referred to as *partially defined games*. In more formal terms, a partially defined game is defined as a triple (N,Z,v) where $N=\{1,2,\ldots,n\}$, Z is a collection of nonempty subsets of N, and v is a real-valued function on Z. We also define a J-game (N,J,v) such that $J \subset N$ and $1,n \in J$ to be a partially defined game where $Z=\{S \subseteq N : |S| \in J\}$. That is, Z is the set of coalitions whose worths are known, while J is the set of cardinalities of the coalitions in Z.

An allocation method for partially defined games

One method of finding a value for partially defined games is to take a geometric approach. We define an *extension* of the partially defined game (N,Z,v) to be a game (N,v') where $v'(S) = v(S)$ for all coalitions $S \in Z$. In other words, the coalitions whose worths are known will receive that worth in the extension. For the other coalitions, their worths can be bounded according to the class of games being considered. This paper considers the zero-monotonic class of games. Therefore, we define the set of zero-monotonic extensions of the J-game v by the following equation:

$$\text{ext}(v) = \{ v' : v'(S) = v(S) \text{ if } |S| \in J, \quad v'(S) \le v'(T) \text{ for all } S \subseteq T \text{ if } |S| \notin J \}$$

The set of extensions for $|S| \in J$ form a convex set $\mathbb{C}$. From this set we can then select some central point and apply the Shapley value to find the allocation.

Coordinate center extension

We first look at a central point called the coordinate center. We define x to be a *coordinate center* of the convex set $\mathbb{C}$ if x_i is the midpoint of $\{ x + \lambda e_i : \lambda \in \mathbb{R} \} \cap \mathbb{C}$ for all i. If $\hat{c}$ is the coordinate center of ext(ω) where ω is a zero-monotonic partially defined game, then $\hat{c} = \{\hat{c}: \hat{c}(S) = \omega(S)$ if $|S| \in J$, $\hat{c}(S) = (1/2)[\max\{ \hat{c}(R) : R \subset S\} + \min\{ \hat{c}(T) : S \subset T\}]$ if $|S| \notin J\}$. By further restricting our study to games where J={1,n-1,n}, the coordinate center extension for the general n-case can be characterized by the following theorem:

<u>Theorem 1</u>: Suppose ω is a zero-normalized, zero-monotonic partially-defined game such that J={1, n - 1, n}. Denote the worths of coalitions S in J such that $|S| \neq 1$ as $\omega(N - \{i\}) = a_i$, with $a_1 \le a_2 \le \ldots \le a_n \le a_0 = \omega(N)$. Let $i(S) = \min\{i : i \notin S\}$.

If $\hat{c}$ is the coordinate center extension of the J-game ω, then:

$$\hat{c}(S) = \omega(S) \quad \text{if } |S| \in J,$$

$$\hat{c}(S) = \frac{s-1}{n-2} a_{i(S)} \quad \text{if } \{1,2,3\} \not\subset S,\ |S| \notin J,$$

$$\hat{c}(S) = \sum_{k=3}^{i(S)-1} \frac{n-s-1}{(n-k)(n-k+1)} a_k + \frac{s-i(S)+2}{n-i(S)+1} a_{i(S)} \quad \text{if } \{1,2,3\} \subseteq S,\ |S| \notin J.$$

Proof: see Appendix A.

Now that we have a generalized formula for the coordinate center extension, we can apply the Shapley value to determine the payoffs to each player. Instead of having to calculate each coalition separately, however, the Shapley value of the coordinate center extension can be simplified as in the following theorem.

Theorem 2 : Let $\Theta = (\Theta_1, \Theta_2, \ldots \Theta_n)$ be the Shapley value allocation for the coordinate center extension of zero-normalized, zero-monotonic partially-defined games where J={1,n -1,n}. That is, $\Theta = \phi(\hat{c})$ where $\hat{c}$ is as given in Theorem 1. Then for $n \geq 4$:

$$\Theta_1 = \frac{1}{n}(a_0 - a_n) + \sum_{k=2}^{n-1} \frac{kn+k+1}{kn(k+1)}(a_{k+1} - a_k) + \frac{n+1}{2n}(a_2 - a_1) + \frac{1}{n}a_1 \ ,$$

$$\Theta_i = \frac{1}{n}(a_0 - a_n) + \sum_{k=i}^{n-1} \frac{kn+k+1}{kn(k+1)}(a_{k+1} - a_k) + \sum_{k=2}^{i-1} \frac{n-k-1}{n(n-k)(k+1)}(a_{k+1} - a_k) + \frac{1}{2n}(a_2 - a_1) + \frac{1}{n}a_1$$

if $i \neq 1$, the first sum is taken to be zero for i=n, and the second sum is taken to be zero for i=2.

Proof: see Appendix B.

An alternate central extension

We now look at a simpler central point that takes the average of the extremal points in each coordinate direction of a set C. We define such a point x, called the coordinate extrema center, as $x_i = ½[\min\{w_i : w \in C\} + \max\{w_i : w \in C\}]$. As with the coordinate center extension, the coordinate extrema center extension can be characterized for the general n-case for games where J={1,n -1,n}.

Theorem 3: Suppose ω is a zero-normalized, zero-monotonic partially-defined game such that

$J=\{1, n-1, n\}$. Denote the worths of coalitions S in J such that $|S| \neq 1$ as $\omega(N - \{i\}) = a_i$, with $a_1 \leq a_2 \leq \ldots \leq a_n \leq a_0 = \omega(N)$. Let $i(S) = \min\{i : i \notin S\}$.

If X is the coordinate extrema center extension of the J-game ω, then:

$X(S) = \omega(S)$ if $|S| \in J$,

$X(S) = \frac{1}{2} a_{i(S)}$ if $|S| \notin J$.

Proof:

Case 1: $|S| \in J$

Follows from definition of extension.

Case 2: $|S| \notin J$

Let $\hat{\omega}$ be a zero-monotonic extension of ω.

First, we have $\min\{\hat{\omega}(S) : \hat{\omega} \in ext(\omega)\} = 0$. Indeed, if $\hat{\omega} \in ext(\omega)$, then $\hat{\omega}(S) \geq \hat{\omega}(\{i\}) = 0$. Further, for the extension $\hat{\omega}$ defined by $\hat{\omega}(R)=0$ if $|R| \notin J$, $\hat{\omega}$ is clearly zero-monotonic and $\hat{\omega}(S) = 0$.

Second, $\max\{\hat{\omega}(S) : \hat{\omega} \in ext(\omega)\} = a_{i(S)}$. If $\hat{\omega} \in ext(\omega)$, $\exists$ $j \notin S$ such that $\hat{\omega}(S) \leq \hat{\omega}(N - \{j\}) \leq \hat{\omega}(N)$. From Case 1, $\hat{\omega}(N - \{j\}) = \omega(N - \{j\}) = a_j$, and from the ordering hypothesis, $\min\{a_j : j \notin S\} = a_{i(S)}$. Thus, $\hat{\omega}(S) \leq a_{i(S)}$. Further, for the extension $\hat{\omega}$ defined by $\hat{\omega}(R) = a_{i(R)}$ if $|R| \notin J$, $\hat{\omega}$ is zero-monotonic and $\hat{\omega}(S) = a_{i(S)}$.

Now by definition, $X(S) = \frac{1}{2}[\min\{\hat{\omega}(S) : \hat{\omega} \in ext(\omega)\} + \max\{\hat{\omega}(S) : \hat{\omega} \in ext(\omega)\}] = \frac{1}{2}[0 + a_{i(S)}] = \frac{1}{2} a_{i(S)}$.

□

Again, rather than calculating X(S) for each coalition separately, the Shapley value allocation for the coordinate extrema center extension can be simplified into a generalized formula.

Theorem 4: Let $\psi=(\psi_1,\psi_2,\ldots,\psi_n)$ be the Shapley value allocation for the coordinate extrema center extension of zero-normalized, zero-monotonic partially-defined games where $J=\{1,n-1,n\}$. That is, $\psi=\phi(X)$ where X is as given in Theorem 3. Then for $n\geq 4$:

$$\psi_1 = \frac{1}{n}(a_0-a_n)+\sum_{k=2}^{n-1}\frac{2n(n-1)-(n-k-1)(n-k)}{2nk(n-1)}(a_{k+1}-a_k)+\frac{n+1}{2n}(a_2-a_1)+\frac{1}{n}a_1 \quad ,$$

$$\psi_i = \frac{1}{n}(a_0-a_n)+\sum_{k=i}^{n-1}\frac{2n(n-1)-(n-k-1)(n-k)}{2nk(n-1)}(a_{k+1}-a_k)+\sum_{k=2}^{i-1}\frac{n-k-1}{2n(n-1)}(a_{k+1}-a_k)+\frac{1}{2n}(a_2-a_1)+\frac{1}{n}a_1$$

if $i\neq 1$, the first sum is taken to be zero for $i=n$, and the second sum is taken to be zero for $i=2$.

Proof: see Appendix C.

Geometric centers

We now focus on the notion of "center"; in particular, consider the coordinate center and coordinate extrema center, as described above, and the centroid. For a simple quadrilateral, as in figure 1, each of these centers coalesce into a single point. This is analagous to zero-monotonic games in which n=4, since there is only one cardinality that is not in the set J. In general, however, these central points need not be identical. Figure 2 illustrates a situation in which each center is distinct.

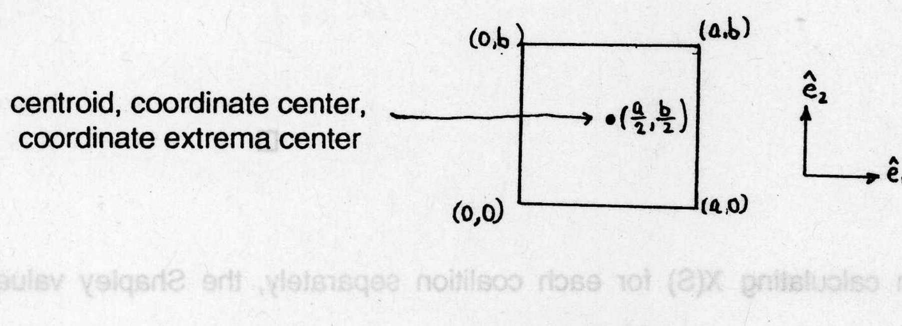

Figure 1

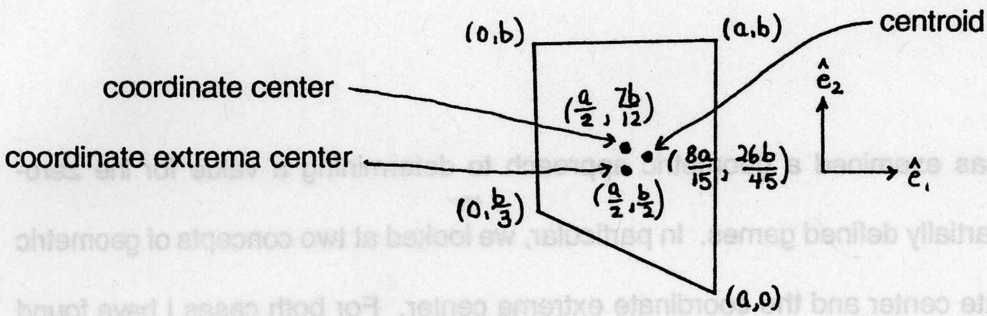

Figure 2

The question arises of which is the "best" definition of center to use in the calculation of the central extension of a partially defined game. While intuitively the centroid appears to be a reasonable choice of center, in practice it becomes complicated to compute for games where n>4. For this reason, we turn to the two centers discussed in this paper. It should be noted that the coordinate center, while easier to calculate, might not always be unique, as demonstrated in figure 3. However, this does not seem to be the case with the zero-monotonic class of games. On the other hand, the coordinate extrema center of a set **C** is not always an element of that set. For example, consider the unit tetrahedron in three-dimensional space. In this case, the coordinate extrema center is (1/2,1/2,1/2), a point which is not an element of the tetrahedron. Nevertheless, it can be shown that the coordinate extrema central extension of the set of zero-monotonic extensions is in fact in the set.

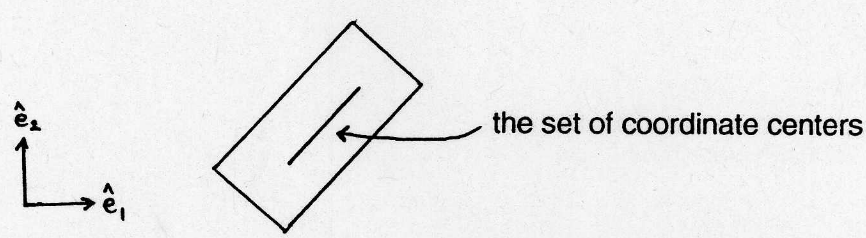

Figure 3

Conclusion

This paper has examined a geometric approach to determining a value for the zero-monotonic class of partially defined games. In particular, we looked at two concepts of geometric center: the coordinate center and the coordinate extrema center. For both cases I have found generalized formulas for the central extension as well as the Shapley value allocation. These results should lead to a more efficient way of allocating savings in situations where a large number of players is involved. However, many more questions still remain open. Future research on this topic could include a comparison of the two types of center presented in this paper, as well as a similar exploration using other centers or classes of games.

References

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Appendix A

In order to prove Theorem 1 we will need the following lemmas:

Lemma 1 If $i \geq 3$, then:

$$\sum_{k=3}^{i-1} \frac{n-s-1}{(n-k)(n-k+1)} + \frac{s-i+2}{n-i+1} = \frac{s-1}{n-2}$$

Proof: We use induction to show:

$$\sum_{k=3}^{i-1} \frac{n-s-1}{(n-k)(n-k+1)} = \frac{s-1}{n-2} - \frac{s-i+2}{n-i+1}$$

Base step: $i=3$

$$\sum_{k=3}^{2} \frac{n-s-1}{(n-k)(n-k+1)} = 0 = \frac{s-1}{n-2} - \frac{s-3+2}{n-3+1}$$

Inductive hypothesis:

Assume that $$\sum_{k=3}^{i-2} \frac{n-s-1}{(n-k)(n-k+1)} = \frac{s-1}{n-2} - \frac{s-(i-1)+2}{n-(i-1)+1}$$

We now show $$\sum_{k=3}^{i-1} \frac{n-s-1}{(n-k)(n-k+1)} = \frac{s-1}{n-2} - \frac{s-i+2}{n-i+1}$$

$$\sum_{k=3}^{i-1} \frac{n-s-1}{(n-k)(n-k+1)} = \sum_{k=3}^{i-2} \frac{n-s-1}{(n-k)(n-k+1)} + \frac{n-s-1}{(n-i+1)(n-i+2)}$$

From the inductive hypothesis, this is equivalent to:

$$= \frac{s-1}{n-2} - \frac{s-(i-1)+2}{n-(i-1)+1} + \frac{n-s-1}{(n-i+1)(n-i+2)}$$

Through algebraic manipulation, this equals:

$$= \frac{s-1}{n-2} - \frac{s-i+2}{n-i+1}.$$

□

Lemma 2 If $\{1,2\} \subseteq R$, then:

$\hat{c}(R) = \sum_{k=3}^{i(R)-1} \frac{n-|R|-1}{(n-k)(n-k+1)} a_k + \frac{|R|-i(R)+2}{n-i(R)+1} a_{i(R)}$ is maximum when $|R|$ and $i(R)$ are maximum, and minimum when $|R|$ and $i(R)$ are minimum.

Proof: Let $c_k = \frac{n-|R|-1}{(n-k)(n-k+1)}$ and $\bar{c}_{i(R)} = \frac{|R|-i(R)+2}{n-i(R)+1}$

(i) Hold $|R|$ constant:

Suppose $i(R)=m$. Then $\hat{c}(R) = \sum_{k=3}^{m-1} c_k a_k + \bar{c}_m a_m$.

It follows from Lemma 1 that $\sum_{k=3}^{i-1} c_k + \bar{c}_i$ is independent of i.

$$\text{Thus, } \sum_{k=3}^{m-1} c_k + \bar{c}_m = \sum_{k=3}^{m} c_k + \bar{c}_{m+1}$$

$$\Rightarrow \sum_{k=3}^{m-1} c_k + \bar{c}_m = \sum_{k=3}^{m-1} c_k + c_m + \bar{c}_{m+1}$$

$$\Rightarrow \bar{c}_m = c_m + \bar{c}_{m+1}$$

Therefore, for $i(R)=m$:

$$\begin{aligned}
\hat{c}(R) &= \sum_{k=3}^{m-1} c_k a_k + c_m a_m + \bar{c}_{m+1} a_m \\
&= \sum_{k=3}^{m} c_k a_k - c_m a_m + c_m a_m + \bar{c}_{m+1} a_m \\
&= \sum_{k=3}^{m} c_k a_k + \bar{c}_{m+1} a_m \\
&\leq \sum_{k=3}^{m} c_k a_k + \bar{c}_{m+1} a_{m+1} = \hat{c}(R) \text{ for } i(R)=m+1
\end{aligned}$$

Hence, for constant $|R|$, $\sum_{k=3}^{i(R)-1} \frac{n-|R|-1}{(n-k)(n-k+1)} a_k + \frac{|R|-i(R)+2}{n-i(R)+1} a_{i(R)}$ is maximum when $i(R)$ is maximum and minimum when $i(R)$ is minimum.

2) Hold $i(R)$ constant:

For $|R|$, we have $\hat{c}(R) = \sum_{K=3}^{i(R)-1} C_K a_K + \bar{C}_{i(R)} a_{i(R)}$.

Suppose we increase $|R|$. Then C_K will decrease by ε_K and $\bar{C}_{i(R)}$ will increase by $\varepsilon_{i(R)}$.

From lemma 1,

$$\sum_{K=3}^{i(R)-1} C_K + \bar{C}_{i(R)} < \sum_{K=3}^{i(R)-1} (C_K - \varepsilon_K) + (\bar{C}_{i(R)} + \varepsilon_{i(R)}) = \sum_{K=3}^{i(R)-1} C_K - \sum_{K=3}^{i(R)-1} \varepsilon_K + \bar{C}_{i(R)} + \varepsilon_{i(R)}$$

$$\Rightarrow 0 < -\sum_{K=3}^{i(R)-1} \varepsilon_K + \varepsilon_{i(R)}$$

$$\Rightarrow 0 > \sum_{K=3}^{i(R)-1} \varepsilon_K - \varepsilon_{i(R)}$$

$$\Rightarrow 0 > \sum_{K=3}^{i(R)-1} \varepsilon_K a_{i(R)} - \varepsilon_{i(R)} a_{i(R)} \geq \sum_{K=3}^{i(R)-1} \varepsilon_K a_K - \varepsilon_{i(R)} a_{i(R)}$$

$$\Rightarrow 0 \geq \sum_{K=3}^{i(R)-1} \varepsilon_K a_K - \varepsilon_{i(R)} a_{i(R)}$$

Then since

$$\sum_{K=3}^{i(R)-1} C_K a_K + \bar{C}_{i(R)} a_{i(R)} = \sum_{K=3}^{i(R)-1} (C_K - \varepsilon_K) a_K + (\bar{C}_{i(R)} + \varepsilon_{i(R)}) a_{i(R)} + \sum_{K=3}^{i(R)-1} \varepsilon_K a_K - \varepsilon_{i(R)} a_{i(R)},$$

we have: $$\sum_{K=3}^{i(R)-1} C_K a_K + \bar{C}_{i(R)} a_{i(R)} \leq \sum_{K=3}^{i(R)-1} (C_K - \varepsilon_K) a_K + (\bar{C}_{i(R)} + \varepsilon_{i(R)}) a_{i(R)} = \hat{c}(R') \text{ for } |R'| > |R|$$

Hence, for constant $i(R)$,

$$\sum_{K=3}^{i(R)-1} \frac{n-|R|-1}{(n-K)(n-K+1)} a_K + \frac{|R|-i(R)+2}{n-i(R)+1} a_{i(R)}$$ is maximum when $|R|$ is maximum and minimum when $|R|$ is minimum.

Note that $|R|$ and $i(R)$ can be simultaneously maximized or minimized. Then,

From (1) and (2), we have now proven the result. $\square$

emma 3 If $S \neq \phi$, then:

$$\sum_{k=3}^{i(S)-2} \frac{n-(|S|-1)-1}{(n-k)(n-k+1)} a_k + \frac{(|S|-1)-i(S)+3}{n-i(S)+2} a_{i(S)-1} = \sum_{k=3}^{i(S)-1} \frac{n-(|S|-1)-1}{(n-k)(n-k+1)} a_k + \frac{(|S|-1)-i(S)+2}{n-i(S)+1} a_{i(S)}.$$

when $i(S) > \max\{i : i \in S\}$.

roof: $i(S) > \max\{i : i \in S\} \Rightarrow i(S) = |S|+1 \Rightarrow |S|-1 = i(S)-2.$

Substitute $i(S)-2$ for $|S|-1$:

$$\sum_{k=3}^{i(S)-2} \frac{n-i(S)+2-1}{(n-k)(n-k+1)} a_k + \frac{i(S)-2-i(S)+3}{n-i(S)+2} a_{i(S)-1} \overset{?}{=} \sum_{k=3}^{i(S)-1} \frac{n-i(S)+2-1}{(n-k)(n-k+1)} a_k + \frac{i(S)-2-i(S)+2}{n-i(S)+1} a_{i(S)}$$

$$\sum_{k=3}^{i(S)-2} \frac{n-i(S)+1}{(n-k)(n-k+1)} a_k + \frac{1}{n-i(S)+2} a_{i(S)-1} \overset{?}{=} \sum_{k=3}^{i(S)-2} \frac{n-i(S)+1}{(n-k)(n-k+1)} a_k + \frac{(n-i(S)+1)}{(n-i(S)+1)(n-i(S)+2)} a_{i(S)-1} + 0$$

$$\frac{1}{n-i(S)+2} a_{i(S)-1} - \frac{1}{n-i(S)+2} a_{i(S)-1} \overset{?}{=} 0$$

$$0 = 0$$

□

Proof of Theorem I:

Outline of proof:

Case I: $|S| \in J$

Proof follows from definition of extension. $\square$

Case II: $|S| \notin J$.

Subcase I: $\{1,2,3\} \not\subseteq S$.

① Show $\max\{\hat{c}(R) : R \subset S\} = \frac{(|S|-1)-1}{n-2} a_{i(S)}$

② Show $\min\{\hat{c}(T) : S \subset T\} = \frac{(|S|+1)-1}{n-2} a_{i(S)}$

③ Show $\hat{c}(S) = \frac{1}{2}[\max\{\hat{c}(R) : R \subset S\} + \min\{\hat{c}(T) : S \subset T\}]$

Subcase 2: $\{1,2,3\} \subseteq S$.

① Show $\max\{\hat{c}(R) : R \subset S\} = \sum_{k=3}^{i(S)-1} \frac{n-(|S|-1)-1}{(n-k)(n-k+1)} a_k + \frac{(|S|-1)-i(S)+2}{n-i(S)+1} a_{i(S)}$

② Show $\min\{\hat{c}(T) : S \subset T\} = \sum_{k=3}^{i(S)-1} \frac{n-(|S|+1)-1}{(n-k)(n-k+1)} a_k + \frac{(|S|+1)-i(S)+2}{n-i(S)+1} a_{i(S)}$

③ Show $\hat{c}(S) = \frac{1}{2}[\max\{\hat{c}(R) : R \subset S\} + \min\{\hat{c}(T) : S \subset T\}]$

ubcase 1:

) Show $\max\{\hat{c}(R) : R \subset S\} = \frac{(|S|-1)-1}{n-2} a_{i(S)}$

Suppose $|S| = 2$.

Then $|R| = 1$, and from case I and zero-normalization, $\max\{\hat{c}(R) : R \subset S\} = 0 = \frac{(|S|-1)-1}{n-2} a_{i(S)}$.

uppose $|S| > 2$.

(i) Let $|R| = 1$

Then from case I and zero-normalization, $\max\{\hat{c}(R) : R \subset S\} = 0 = \frac{|R|-1}{n-2} a_{i(R)}$.

(ii) Let $|R| \neq 1$.

$\{1,2,3\} \not\subset S \Rightarrow \{1,2,3\} \not\subset R \Rightarrow \hat{c}(R) = \frac{|R|-1}{n-2} a_{i(R)}$.

rom (i) and (ii), we have
$$\max\{\hat{c}(R) : R \subset S\} = \max\left\{\frac{|R|-1}{n-2} a_{i(R)} : R \subset S\right\}$$
$$\leq \max\left\{\frac{|R|-1}{n-2} : R \subset S\right\} \max\{a_{i(R)} : R \subset S\}$$

(a) $\max\left\{\frac{|R|-1}{n-2} : R \subset S\right\}$:

$\frac{|R|-1}{n-2}$ is maximum when $|R|$ is maximum

$R \subset S \Rightarrow |R| < |S| \Rightarrow \max\{|R|\} = |S|-1$

$\Rightarrow \max\left\{\frac{|R|-1}{n-2} : R \subset S\right\} = \frac{(|S|-1)-1}{n-2}$

(b) $\max\{a_{i(R)} : R \subset S\}$:

By definition, $\max\{a_{i(R)} : R \subset S\} = \max\{w(N - i(R)) : R \subset S\}$.

It follows from the ordering hypothesis that if $w(N-\{r\}) \geq w(N-\{s\})$, then $r > s$.

Therefore, $w(N-i(R))$ is maximum when $i(R)$ is maximum.

$$R \subset S \Rightarrow \min\{i : i \notin R\} \leq \min\{i : i \notin S\}$$

$$\Rightarrow i(R) \leq i(S)$$

$$\Rightarrow \max\{i(R)\} \leq i(S)$$

Thus, $\max\{w(N-i(R)) : R \subset S\} \leq w(N-i(S)) = a_{i(S)}$.

$$\Rightarrow \max\{a_{i(R)} : R \subset S\} \leq a_{i(S)}.$$

It follows from (a) and (b) that

$$\max\{\hat{c}(R) : R \subset S\} \leq \max\left\{\frac{|R|-1}{n-2} : R \subset S\right\} \max\{a_{i(R)} : R \subset S\} \leq \frac{(|S|-1)-1}{n-2} a_{i(S)}.$$

Now exhibit an $R \subset S$ such that $\hat{c}(R) = \dfrac{(|S|-1)-1}{n-2} a_{i(S)}$:

Since $\{1,2,3\} \not\subset S$, it follows that $i(S) = 1, 2, \text{or } 3$.

$|S| > 2$, so $\exists\, j \in S$ such that $j > i(S)$.

Let $R = S - j$.

Then $|R| = |S| - 1$ and $i(R) = i(S)$.

$$\Rightarrow \hat{c}(R) = \frac{|R|-1}{n-2} a_{i(R)} = \frac{(|S|-1)-1}{n-2} a_{i(S)}.$$

We now have $\max\{\hat{c}(R) : R \subset S\} = \dfrac{(|S|-1)-1}{n-2} a_{i(S)}$.

□

2) Show $\min\{\hat{c}(T): S \subset T\} = \frac{(|S|+1)-1}{n-2} a_{i(S)}$.

Suppose $|S| = n-2$.

Then $|T| = n-1$ or n.

Let $|T| = n$.

Then from case I and hypothesis, $\hat{c}(T) = w(T) = a_0$.

Let $|T| = n-1$.

Then $\hat{c}(T) = w(N - \{i\}) = a_i$, where $i \notin T$.

By ordering hypothesis, $a_0 \geq a_i$, so $\hat{c}(T)$ is minimum when $|T| = n-1$.

$\Rightarrow \min\{\hat{c}(T): S \subset T\} = \min\{a_i : i \notin T\} = \min\{a_{i(T)}: S \subset T\}$

Moreover, it also follows from the ordering hypothesis that $a_{i(T)}$ is minimum when $i(T)$ is minimum.

$S \subset T \Rightarrow \min\{i : i \notin S\} \leq \min\{i : i \notin T\}$

$\Rightarrow i(S) \leq i(T)$

$\Rightarrow \min i(T) \geq i(S)$

$\Rightarrow \min\{\hat{c}(T): S \subset T\} = \min\{a_{i(T)}: S \subset T\} \geq a_{i(S)}$

Now exhibit a $T \supset S$ such that $\hat{c}(T) = a_{i(S)}$:

$|S| = n-2 \Rightarrow \exists$ two elements not in S, $i(S)$ and j.

Let $T = S \cup \{j\}$. Then $T = N - i(S)$. By case I, $\hat{c}(T) = w(N - i(S)) = a_{i(S)}$.

We now have $\min\{\hat{c}(T): S \subset T\} = a_{i(S)} = \frac{(|S|+1)-1}{n-2} a_{i(S)}$.

suppose $|S| < n-2$ and $T \supset S$.

(i) Let $|T| \geq n-1$.

Following above argument, $\min\{\hat{c}(T) : S \subset T\} = a_{i(S)} = \frac{|T|-1}{n-2} a_{i(T)}$.

(ii) Let $|T| < n-1$.

Then (1) $\hat{c}(T) = \frac{|T|-1}{n-2} a_{i(T)}$ if $T \not\supseteq \{1,2,3\}$

or

(2) $\hat{c}(T) = \sum_{K=3}^{i(T)-1} \frac{n-|T|-1}{(n-k)(n-k+1)} a_K + \frac{|T|-i(T)+2}{n-i(T)+1} a_{i(T)}$ if $T \supseteq \{1,2,3\}$

If $v' \not\supseteq \{1,2,3\}$ and $v \supseteq \{1,2,3\}$, then it follows from lemma 1 that for $|v'| = |v|$:

$$\frac{|v'|-1}{n-2} = \sum_{K=3}^{i(v)-1} \frac{n-|v|-1}{(n-k)(n-k+1)} + \frac{|v|-i(v)+2}{n-i(v)+1}$$

$$\Rightarrow \frac{|v'|-1}{n-2} a_{i(v')} = \sum_{K=3}^{i(v)-1} \frac{n-|v|-1}{(n-k)(n-k+1)} a_{i(v')} + \frac{|v|-i(v)+2}{n-i(v)+1} a_{i(v')}$$

$v' \not\supseteq \{1,2,3\} \Rightarrow i(v') \leq 3$

$v \supseteq \{1,2,3\} \Rightarrow i(v) > 3$

$\Rightarrow i(v') < i(v)$

$\Rightarrow a_{i(v')} \leq a_{i(v)}$

$$\Rightarrow \frac{|v|-i(v)+2}{n-i(v)+1} a_{i(v')} \leq \frac{|v|-i(v)+2}{n-i(v)+1} a_{i(v)}$$

Moreover, $i(v) > 3 \Rightarrow i(v)-1 \geq 3$

$$\Rightarrow \sum_{K=3}^{i(v)-1} \frac{n-|v|-1}{(n-k)(n-k+1)} a_{i(v')} \leq \sum_{K=3}^{i(v)-1} \frac{n-|v|-1}{(n-k)(n-k+1)} a_K$$

Therefore, $\frac{|v'|-1}{n-2} a_{i(v')} \leq \sum_{K=3}^{i(v)-1} \frac{n-|v|-1}{(n-k)(n-k+1)} a_K + \frac{|v|-i(v)+2}{n-i(v)+1} a_{i(v)}$

It now follows that $(1) \leq (2)$.

Thus, $\min\{\hat{c}(T): S\subset T\} = \min\{\frac{|T|-1}{n-2} a_{i(T)}: S\subset T\}$

From (i) and (ii), we have $\min\{\hat{c}(T): S\subset T\} = \min\{\frac{|T|-1}{n-2} a_{i(T)}: S\subset T\}$

$$\geq \min\left\{\frac{|T|-1}{n-2}: S\subset T\right\} \min\{a_{i(T)}: S\subset T\}$$

(a) $\min\left\{\frac{|T|-1}{n-2}: S\subset T\right\}$:

$\frac{|T|-1}{n-2}$ is minimum when $|T|$ is minimum.

$$S\subset T \Rightarrow |S|<|T| \Rightarrow \min\{|T|\} = |S|+1$$

$$\Rightarrow \min\left\{\frac{|T|-1}{n-2}: S\subset T\right\} = \frac{(|S|+1)-1}{n-2}$$

(b) $\min\{a_{i(T)}: S\subset T\}$:

By definition, $\min\{a_{i(T)}: S\subset T\} = \min\{w(N-i(T)): S\subset T\}$.

It follows from the ordering hypothesis that if $w(N-\{r\}) \geq w(N-\{s\})$, then $r>s$. Thus, $w(N-i(T))$ is minimum when $i(T)$ is minimum

$$S\subset T \Rightarrow \min\{i: i\notin S\} \leq \min\{i: i\notin T\}$$

$$\Rightarrow i(S) \leq i(T)$$

$$\Rightarrow i(S) \leq \min\{i(T)\}$$

Therefore, $\min\{w(N-i(T)): S\subset T\} \geq w(N-i(S)) = a_{i(S)}$

$$\Rightarrow \min\{a_{i(T)}: S\subset T\} \geq a_{i(S)}.$$

It follows from (a) and (b) that

$$\min\{\hat{c}(T): S\subset T\} \geq \min\left\{\frac{|T|-1}{n-2}: S\subset T\right\} \min\{a_{i(T)}: S\subset T\} \geq \frac{(|S|+1)-1}{n-2} a_{i(S)}$$

Now exhibit a $T \supset S$ such that $\hat{c}(T) = \frac{(|S|+1)-1}{n-2} a_{i(S)}$:

Since $|S| < n-2$, it follows that $\exists$ at least three elements that are not in S, $i(S), j, k$, where $i(S) < j < k$.

Let $T = S \cup \{k\}$.

Then $|T| = |S|+1$ and $i(T) = i(S)$.

$S \not\supset \{1,2,3\} \Rightarrow$ if $k \notin \{1,2,3\}$, then $T \not\supset \{1,2,3\}$

if $k \in \{1,2,3\}$, then $j \in \{1,2,3\} \Rightarrow T \not\supset \{1,2,3\}$.

Thus $\hat{c}(T) = \frac{|T|-1}{n-2} a_{i(T)} = \frac{(|S|+1)-1}{n-2} a_{i(S)}$.

We now have $\min\{\hat{c}(T): S \subset T\} = \frac{(|S|+1)-1}{n-2} a_{i(S)}$

$\square$

2) Show $\hat{c}(S) = \frac{1}{2}[\max\{\hat{c}(R): R \subset S\} + \min\{\hat{c}(T): S \subset T\}]$

From ①, $\max\{\hat{c}(R): R \subset S\} = \frac{(|S|-1)-1}{n-2} a_{i(S)}$ and from ②, $\min\{\hat{c}(T): S \subset T\} = \frac{(|S|+1)-1}{n-2} a_{i(S)}$.

Thus $\frac{1}{2}[\max\{\hat{c}(R): R \subset S\} + \min\{\hat{c}(T): S \subset T\}]$

$$= \frac{1}{2}\left[\frac{(|S|-1)-1}{n-2} a_{i(S)} + \frac{(|S|+1)-1}{n-2} a_{i(S)}\right]$$

$$= \frac{1}{2}\left[\frac{(|S|-1)-1+(|S|+1)-1}{n-2}\right] a_{i(S)}$$

$$= \frac{1}{2}\left[\frac{2|S|-2}{n-2}\right] a_{i(S)}$$

$$= \frac{|S|-1}{n-2} a_{i(S)}$$

$$= \hat{c}(S).$$

$\square$

Subcase 2:

① Show $\max\{\hat{c}(R): R \subset S\} = \sum_{k=3}^{i(S)-1} \frac{n-(|S|-1)-1}{(n-k)(n-k+1)} a_k + \frac{(|S|-1)-i(S)+2}{n-i(S)+1} a_{i(S)}$

Suppose $|S| = 3$.

(i) Let $|R| = 1$.

Then from case I and zero-normalization, $\max\{\hat{c}(R): R \subset S\} = 0$.

(ii) Let $|R| = 2$.

Then $\{1,2,3\} \not\subset R \Rightarrow \hat{c}(R) = \frac{|R|-1}{n-2} a_{i(R)} = \frac{2-1}{n-2} a_{i(R)} = \frac{1}{n-2} a_{i(R)}$

$i(R) = 1, 2$, or 3 and $a_1 \le a_2 \le a_3$.

$$\Rightarrow \max\{\hat{c}(R): R \subset S\} = \frac{1}{n-2} a_3$$

$$= \frac{n-3}{(n-3)(n-2)} a_3 + 0$$

$$= \sum_{k=3}^{3} \frac{n-2-1}{(n-k)(n-k+1)} a_k + \frac{2-4+2}{n-4+1} a_4$$

$$= \sum_{k=3}^{i(S)-1} \frac{n-(|S|-1)-1}{(n-k)(n-k+1)} a_k + \frac{(|S|-1)-i(S)+2}{n-i(S)+1} a_{i(S)}.$$

From (i) and (ii), $\max\{\hat{c}(R): R \subset S\} = 0$ or

$$\max\{\hat{c}(R): R \subset S\} = \sum_{k=3}^{i(S)-1} \frac{n-(|S|-1)-1}{(n-k)(n-k+1)} a_k + \frac{(|S|-1)-i(S)+2}{n-i(S)+1} a_{i(S)}$$

Since $\sum_{k=3}^{i(S)-1} \frac{n-(|S|-1)-1}{(n-k)(n-k+1)} a_k + \frac{(|S|-1)-i(S)+2}{n-i(S)+1} a_{i(S)} \ge 0$, it follows that

$$\max\{\hat{c}(R): R \subset S\} = \sum_{k=3}^{i(S)-1} \frac{n-(|S|-1)-1}{(n-k)(n-k+1)} a_k + \frac{(|S|-1)-i(S)+2}{n-i(S)+1} a_{i(S)}.$$

Suppose $|S|>3$.

Then (1) $\hat{c}(R)=\frac{|R|-1}{n-2}a_{i(R)}$ if $R \not\supseteq \{1,2,3\}$

or

(2) $\hat{c}(R)=\sum_{k=3}^{i(R)-1}\frac{n-|R|-1}{(n-k)(n-k+1)}a_k+\frac{|R|-i(R)+2}{n-i(R)+1}a_{i(R)}$ if $\{1,2,3\}\subseteq R$

From previous argument, (1) $\leq$ (2).

Therefore,

$$\max\{\hat{c}(R):R\subset S\}=\max\left\{\sum_{k=3}^{i(R)-1}\frac{n-|R|-1}{(n-k)(n-k+1)}a_k+\frac{|R|-i(R)+2}{n-i(R)+1}a_{i(R)}:R\supseteq\{1,2,3\},R\subset S\right\}$$

From Lemma 2, $\hat{c}(R)$ is maximum when $|R|$ and $i(R)$ are maximum.

$R\subset S \Rightarrow |R|<|S| \Rightarrow \max\{|R|\}=|S|-1$.

Moreover, $R\subset S \Rightarrow \min\{i: i\notin R\}\leq \min\{i:i\notin S\}$

$\Rightarrow i(R)\leq i(S)$

$\Rightarrow \max\{i(R)\}\leq i(S)$.

$$\Rightarrow \max\{\hat{c}(R):R\subset S\}\leq \sum_{k=3}^{i(S)-1}\frac{n-(|S|-1)-1}{(n-k)(n-k+1)}a_k+\frac{(|S|-1)-i(S)+2}{n-i(S)+1}a_{i(S)}.$$

Now exhibit an $R\subset S$ such that

$$\hat{c}(R)=\sum_{k=3}^{i(S)-1}\frac{n-(|S|-1)-1}{(n-k)(n-k+1)}a_k+\frac{(|S|-1)-i(S)+2}{n-i(S)+1}a_{i(S)}:$$

let $R=S-\max\{i:i\in S\}$

Then $|R|=|S|-1$.

$S\supseteq\{1,2,3\}$, $|S|>3 \Rightarrow \max\{i:i\in S\}\notin\{1,2,3\} \Rightarrow R\supseteq\{1,2,3\}$.

Case 1: Suppose $i(S)<\max\{i:i\in S\}$.

Then $i(R)=i(S)$

$$\Rightarrow \hat{c}(R)=\sum_{k=3}^{i(S)-1}\frac{n-(|S|-1)-1}{(n-k)(n-k+1)}a_k+\frac{(|S|-1)-i(S)+2}{n-i(S)+1}a_{i(S)}.$$

case 2: Suppose $i(S) > \max\{i : i \in S\}$

Then $i(R) = \max\{i : i \in S\} = i(S) - 1$.

$$\Rightarrow \hat{c}(R) = \sum_{k=3}^{i(S)-1-1} \frac{n-(|S|-1)-1}{(n-k)(n-k+1)} a_k + \frac{(|S|-1)-(i(S)-1)+2}{n-(i(S)-1)+1} a_{i(S)-1}$$

(by Lemma 3)
$$= \sum_{k=3}^{i(S)-1} \frac{n-(|S|-1)-1}{(n-k)(n-k+1)} a_k + \frac{(|S|-1)-i(S)+2}{n-i(S)+1} a_{i(S)}.$$

We now have
$$\max\{\hat{c}(R) : R \subset S\} = \sum_{k=3}^{i(S)-1} \frac{n-(|S|-1)-1}{(n-k)(n-k+1)} a_k + \frac{(|S|-1)-i(S)+2}{n-i(S)+1} a_{i(S)}$$

$\square$

② Show
$$\min\{\hat{c}(T) : S \subset T\} = \sum_{k=3}^{i(S)-1} \frac{n-(|S|+1)-1}{(n-k)(n-k+1)} a_k + \frac{(|S|+1)-i(S)+2}{n-i(S)+1} a_{i(S)}$$

Suppose $|S| = n-2$.

Then $|T| = n-1$ or n.

Let $|T| = n$.

Then from case I and hypothesis, $\hat{c}(T) = \omega(T) = a_0$.

Let $|T| = n-1$.

Then $\hat{c}(T) = \omega(N - \{i\}) = a_i$, where $i \notin T$

By ordering hypothesis, $a_0 \geq a_i$, so $\hat{c}(T)$ is minimum when $|T| = n-1$.

$$\Rightarrow \min\{\hat{c}(T) : S \subset T\} = \min\{a_i : i \notin T\} = \min\{a_{i(T)} : S \subset T\}$$

Moreover, it also follows from the ordering hypothesis that $a_{i(T)}$ is minimum when $i(T)$ is minimum

$$S \subset T \Rightarrow \min\{i : i \notin S\} \leq \min\{i : i \notin T\}$$
$$\Rightarrow i(S) \leq i(T) \Rightarrow \min i(T) \geq i(S)$$

$$\Rightarrow \min\{\hat{c}(T) : S \subset T\} = \min\{a_{i(T)} : S \subset T\} \geq a_{i(S)}$$

Now exhibit a $T \supset S$ such that $\hat{c}(T) = a_{i(S)}$:

$|S| = n-2 \Rightarrow \exists$ two elements not in S, $i(S)$ and j.

Let $T = S \cup \{j\}$. Then $T = N - i(S)$.

By case I, $\hat{c}(T) = \omega(N - i(S)) = a_{i(S)}$.

$$\Rightarrow \min\{\hat{c}(T) : S \subset T\} = a_{i(S)} = \sum_{k=3}^{i(S)-1} \frac{n-(|S|+1)-1}{(n-k)(n-k+1)} a_k + \frac{(|S|+1)-i(S)+2}{n-i(S)+1} a_{i(S)}.$$

Suppose $|S| < n-2$.

(i) Let $|T| \geq n-1$.

Following above argument, $\min\{\hat{c}(T) : S \subset T\} = a_{i(S)} = \sum_{k=3}^{i(T)-1} \frac{n-|T|-1}{(n-k)(n-k+1)} a_k + \frac{|T|-i(T)+2}{n-i(T)+1} a_{i(T)}$

(ii) Let $|T| < n-1$.

$S \supseteq \{1,2,3\} \Rightarrow T \supseteq \{1,2,3\}$

$$\Rightarrow \hat{c}(T) = \sum_{k=3}^{i(T)-1} \frac{n-|T|-1}{(n-k)(n-k+1)} a_k + \frac{|T|-i(T)+2}{n-i(T)+1} a_{i(T)}$$

Thus, $\min\{\hat{c}(T) : S \subset T\} = \min\left\{\sum_{k=3}^{i(T)-1} \frac{n-|T|-1}{(n-k)(n-k+1)} a_k + \frac{|T|-i(T)+2}{n-i(T)+1} a_{i(T)} : S \subset T\right\}$

From (i) and (ii), we have

$$\min\{\hat{c}(T) : S \subset T\} = \min\left\{\sum_{k=3}^{i(T)-1} \frac{n-|T|-1}{(n-k)(n-k+1)} a_k + \frac{|T|-i(T)+2}{n-i(T)+1} a_{i(T)} : S \subset T\right\}$$

From Lemma 2, $\hat{c}(T)$ is minimum when $|T|$ and $i(T)$ are minimum.

$S \subset T \Rightarrow |S| < |T| \Rightarrow \min\{|T|\} = |S|+1$

Moreover, $S \subset T \Rightarrow \min\{i : i \notin S\} \leq \min\{i : i \notin T\}$

$\Rightarrow i(S) \leq i(T)$

$\Rightarrow i(S) \leq \min\{i(T)\}$

$$\Rightarrow \min\{\hat{c}(T) : S \subset T\} \geq \sum_{k=3}^{i(S)-1} \frac{n-(|S|+1)-1}{(n-k)(n-k+1)} a_k + \frac{(|S|+1)-i(S)+2}{n-i(S)+1} a_{i(S)}$$

Now exhibit a $T \supset S$ such that

$$\hat{c}(T) = \sum_{k=3}^{i(S)-1} \frac{n-(|S|+1)-1}{(n-k)(n-k+1)} a_k + \frac{(|S|+1)-i(S)+2}{n-i(S)+1} a_{i(S)} :$$

Let $T = S \cup \max\{i : i \notin S\}$.

Then $T \supset \{1,2,3\}$ and $|T| = |S|+1$.

Moreover, $|S| < n-1$, so $\exists$ at least two elements that are not in S

$\Rightarrow i(S) < \max\{i : i \notin S\}$

$\Rightarrow i(S) = i(T)$.

Thus, $$\hat{c}(T) = \sum_{k=3}^{i(S)-1} \frac{n-(|S|+1)-1}{(n-k)(n-k+1)} a_k + \frac{(|S|+1)-i(S)+2}{n-i(S)+1} a_{i(S)}$$

We now have $$\min\{\hat{c}(T) : S \subset T\} = \sum_{k=3}^{i(S)-1} \frac{n-(|S|+1)-1}{(n-k)(n-k+1)} a_k + \frac{(|S|+1)-i(S)+2}{n-i(S)+1} a_{i(S)}.$$

$\square$

③ Show $\hat{c}(S) = \frac{1}{2}\left[\max\{\hat{c}(R) : R \subset S\} + \min\{\hat{c}(T) : S \subset T\}\right]$

From ①, $$\max\{\hat{c}(R) : R \subset S\} = \sum_{k=3}^{i(S)-1} \frac{n-(|S|-1)-1}{(n-k)(n-k+1)} a_k + \frac{(|S|-1)-i(S)+2}{n-i(S)+1} a_{i(S)}$$

From ②, $$\min\{\hat{c}(T) : S \subset T\} = \sum_{k=3}^{i(S)-1} \frac{n-(|S|+1)-1}{(n-k)(n-k+1)} a_k + \frac{(|S|+1)-i(S)+2}{n-i(S)+1} a_{i(S)}.$$

Thus, $\frac{1}{2}\left[\max\{\hat{c}(R) : R \subset S\} + \min\{\hat{c}(T) : S \subset T\}\right]$

$$= \frac{1}{2}\left[\sum_{k=3}^{i(S)-1} \frac{n-(|S|-1)-1}{(n-k)(n-k+1)} a_k + \frac{(|S|-1)-i(S)+2}{n-i(S)+1} a_{i(S)} + \sum_{k=3}^{i(S)-1} \frac{n-(|S|+1)-1}{(n-k)(n-k+1)} a_k + \frac{(|S|+1)-i(S)+2}{n-i(S)+1} a_{i(S)}\right]$$

$$= \frac{1}{2}\left[\sum_{k=3}^{i(S)-1} \frac{n-|S|+1-1+n-|S|-1-1}{(n-k)(n-k+1)} a_k + \frac{|S|-1-i(S)+2+|S|+1-i(S)+2}{n-i(S)+1} a_{i(S)}\right]$$

$$= \frac{1}{2}\left[\sum_{k=3}^{i(S)-1} \frac{2n-2|S|-2}{(n-k)(n-k+1)} a_k + \frac{2|S|-2i(S)+4}{n-i(S)+1} a_{i(S)}\right]$$

$$= \sum_{k=3}^{i(S)-1} \frac{n-|S|-1}{(n-k)(n-k+1)} a_k + \frac{|S|-i(S)+2}{n-i(S)+1} a_{i(S)}$$

$$= \hat{c}(S).$$

$\square$

ppendix B

The following lemma will be necessary in the proof of Theorem 2:

mma 1 If $n > m \geq 1$, then:

$$\frac{1}{n} + \frac{n-m+1}{n(n-1)} + \frac{(n-m)!}{n!}\sum_{s=m-1}^{n-2}\frac{(s-m+2)(s-1)!}{(s-m+1)!} = \frac{n(m-1)+m}{mn(m-1)}$$

of: We use induction on n to show

$$\frac{2n-m}{n(n-1)} - \frac{n(m-1)+m}{mn(m-1)} + \frac{(n-m)!}{n!}\sum_{s=m-1}^{n-2}\frac{(s-m+2)(s-1)!}{(s-m+1)!} = 0$$

Base case: $n = m+1$:

$$\frac{2m+2-m}{m(m+1)} - \frac{(m+1)(m-1)+m}{m(m+1)(m-1)} + \frac{1}{(m+1)!}\frac{(m-2)!}{1} \stackrel{?}{=} 0$$

$$\frac{(m+2)(m-1)-(m+1)(m-1)-m}{m(m+1)(m-1)} + \frac{1}{m(m+1)(m-1)} = 0$$

$$\frac{(m-1)(m+2)-(m-1)(m+1)-m+1}{m(m-1)(m+1)} \stackrel{?}{=} 0$$

$$0 = 0.$$

Inductive hypothesis:

Assume $$\frac{2n-m}{n(n-1)} - \frac{n(m-1)+m}{mn(m-1)} + \frac{(n-m)!}{n!}\sum_{s=m-1}^{n-2}\frac{(s-m+2)(s-1)!}{(s-m+1)!} = 0$$

We show: $$\frac{2(n+1)-m}{(n+1)n} - \frac{(n+1)(m-1)+m}{m(m-1)(n+1)} + \frac{(n-m+1)!}{(n+1)!}\sum_{s=m-1}^{n-1}\frac{(s-m+2)(s-1)!}{(s-m+1)!} = 0$$

$$\frac{2n-m+2}{n(n+1)} - \frac{(n+1)(m-1)+m}{m(m-1)(n+1)} + \frac{(n-m+1)!}{(n+1)!}\left[\sum_{s=m-1}^{n-2}\frac{(s-m+2)(s-1)!}{(s-m+1)!} + \frac{(n-m+1)(n-2)!}{(n-m)!}\right] \stackrel{?}{=} 0$$

By the inductive hypothesis, we have:

$$\frac{2n-m+2}{n(n+1)} - \frac{(n+1)(m-1)+m}{m(m-1)(n+1)} + \frac{(n-m+1)!}{(n+1)!}\frac{(n-m+1)(n-2)!}{(n-m)!} + \frac{(n-m+1)!}{(n+1)!}\frac{n!}{(n-m)!}\left[\frac{n(m-1)+m}{mn(m-1)} - \frac{2n-m}{n(n-1)}\right] \stackrel{?}{=} 0$$

$$\frac{2n-m+2}{n(n+1)} - \frac{(n+1)(m-1)+m}{m(m-1)(n+1)} + \frac{(n-m+1)(n-m+1)}{(n-1)(n+1)n} + \frac{(n-m+1)}{(n+1)}\left[\frac{n(m-1)+m}{mn(m-1)} - \frac{2n-m}{n(n-1)}\right] \stackrel{?}{=} 0$$

$$\frac{2n-m+2}{n(n+1)} - \frac{(n+1)(m-1)+m}{m(m-1)(n+1)} + \frac{(n-m+1)(n-m+1)}{n(n-1)(n+1)} + \frac{(n-m+1)(n(m-1)+m)}{mn(m-1)(n+1)} - \frac{(n-m+1)(2n-m)}{n(n-1)(n+1)} \stackrel{?}{=} 0$$

$$\frac{(n-m+1)(mn-n+m)}{mn(m-1)(n+1)} - \frac{n(n+1)(m-1)+mn}{mn(m-1)(n+1)} + \frac{(n-m+1)(n-m+1-2n+m)}{n(n-1)(n+1)} + \frac{2n-m+2}{n(n+1)} \stackrel{?}{=} 0$$

$$\frac{2n-m+2-n-1}{n(n+1)} + \frac{(n-m+1)(1-n)}{n(n-1)(n+1)} \stackrel{?}{=} 0$$

$$\frac{2n-m+2-n-1-n+m-1}{n(n+1)} \stackrel{?}{=} 0$$

$$0=0$$

□

roof of Theorem 2:

Note that $\hat{c}$ is linear in a_j while the Shapley value is linear in $\hat{c}$. Therefore, θ is linear in a_j.
It follows that θ can be uniquely determined on a basis for the a_j's.

We first find $\varphi(\hat{c})$ for the following basis:

$a_0=1,\ a_n=\dots=a_1=0$

$a_0=a_n=1,\ a_{n-1}=\dots=a_1=0$

$a_0=a_n=\dots=a_m=1,\ a_{m-1}=\dots=a_1=0\ ,(n>m>2)$

$a_0=a_n=\dots=a_2=1,\ a_1=0$

$a_0=a_n=\dots=a_1=1$

ase 1: Let $a_0=1,\ a_n=\dots=a_1=0$.
Then it follows from symmetry that each player should receive the same payoff.
By efficiency, we have $\varphi_i(\hat{c})=\frac{1}{n}$.

ase 2: Let $a_0=a_n=1,\ a_{n-1}=\dots=a_1=0$
For $|S|<n-1$, $i(S)\le n-1 \Rightarrow a_{i(S)}=0$. Therefore, $\hat{c}(S)=\begin{cases}1, & S=N,\ N-\{n\}\\ 0, & \text{otherwise}\end{cases}$
It then follows that $\varphi_{i<n}(\hat{c})=\frac{1}{n}(1-0)+\frac{1}{n(n-1)}(1-0)=\frac{1}{n-1}$

$\varphi_n(\hat{c})=\frac{1}{n}(1-1)=0$

ase 3: Let $a_0 = a_n = \ldots = a_m = 1$, $a_{m-1} = \ldots = a_1 = 0$, $(n > m > 2)$

We only need to compute the Shapley value of $\hat{c}$ for coalitions where the player's marginal contribution is nonzero.

By Lemma 1 of Appendix A, Theorem 1 can be simplified to: $\hat{c}(S) = \omega(S)$ if $|S| \in J$,

$$\hat{c}(S) = \frac{s-1}{n-2} - \sum_{k=3}^{m-1} \frac{n-s-1}{(n-k)(n-k+1)} \text{ if } |S| \notin J.$$

Then there are four possible types of nonzero marginal contributions: $1-0$, $\left(\frac{s-1}{n-2} - \sum_{k=3}^{m-1} \frac{n-s-1}{(n-k)(n-k+1)}\right) - 0$,

$1 - \left(\frac{s-2}{n-2} - \sum_{k=3}^{m-1} \frac{n-s}{(n-k)(n-k+1)}\right)$, or $\left(\frac{s-1}{n-2} - \sum_{k=3}^{m-1} \frac{n-s-1}{(n-k)(n-k+1)}\right) - \left(\frac{s-2}{n-2} - \sum_{k=3}^{m-1} \frac{n-s}{(n-k)(n-k+1)}\right)$.

Subcase 1: Player's marginal contribution equals $1-0$.

There are four ways in which this can occur:

① $S = N$, $S - \{i\} \not\ni k$ for some $k < m$.

This is only possible for $i < m$. The Shapley value is then $\frac{1}{n}(1-0)$.

② $S = N$, $|S - \{i\}| = 1$

Since $n \geq 4$, this case is impossible.

③ $S = N - \{j\}$ for some $j \geq m$, $S - \{i\} \not\ni k$ for some $k < m$

This can only occur when $i < m$. There are $n-m+1$ such coalitions S that contain i, and so the Shapley value contribution is then $(n-m+1)\frac{1}{n(n-1)}(1-0)$.

④ $S = N - \{j\}$, $j \geq m$, $|S - \{i\}| = 1$.

Since $n \geq 4$, this case is impossible.

Subcase 2: Player's marginal contribution equals $1 - \left(\frac{s-2}{n-2} - \sum_{k=3}^{m-1} \frac{n-s}{(n-k)(n-k+1)}\right)$

To obtain $\hat{c}(S) = 1$, we can have $S = N$ or $S = N - \{j\}$, $j \geq m$.

But in order to obtain $\hat{c}(S - \{i\}) = \frac{s-2}{n-2} - \sum_{k=3}^{m-1} \frac{n-s}{(n-k)(n-k+1)}$, we must have $1 < |S - \{i\}| < n-1$ with $\{1, \ldots, m-1\} \subseteq S - \{i\}$. $|S|$ must then equal $n-1$, and therefore $S = N - \{j\}$ for some $j \geq m$.

For $i<m$, these conditions lead to a contradiction. For $i \geq m$, there are $n-m$ such coalitions that contain i, and so the Shapley value contribution is $(n-m)\frac{1}{n(n-1)}\left(1-\left(\frac{n-3}{n-2}-\sum_{k=3}^{m-1}\frac{1}{(n-k)(n-k+1)}\right)\right)$.

Subcase 3: Player's marginal contribution equals $\left(\frac{s-1}{n-2}-\sum_{k=3}^{m-1}\frac{n-s-1}{(n-k)(n-k+1)}\right)-0$

For this to occur, we must have $\{1,\cdots,m-1\}\subseteq S$, with $m-1\leq |S|<n-1$ and either $S-\{i\}\not\ni k$ for some $k<m$, or $|S-\{i\}|=1$.

① $S-\{i\}\not\ni k$, $k<m$.

This is only possible for $i<m$. Then we want $S=\{1,2,\ldots,m-1\}\cup R$, where $R\subseteq\{m,m+1,\cdots,n\}$ and $|R|=s-m+1$. There are $\binom{n-m+1}{s-m+1}$ such coalitions for each cardinality. Thus, the Shapley value contribution is

$$\sum_{s=m-1}^{n-2}\binom{n-m+1}{s-m+1}\frac{(s-1)!(n-s)!}{n!}\left[\left(\frac{s-1}{n-2}-\sum_{k=3}^{m-1}\frac{n-s-1}{(n-k)(n-k+1)}\right)-0\right]$$

② $|S-\{i\}|=1$

This implies $|S|=2$. If $i<m$, then this case is the same as ①. If $i\geq m$, then this case is impossible, since $S\supseteq\{1,\cdots,m-1,i\}$ and therefore $|S|>2$.

Subcase 4: Player's marginal contribution equals $\left(\frac{s-1}{n-2}-\sum_{k=3}^{m-1}\frac{n-s-1}{(n-k)(n-k+1)}\right)-\left(\frac{s-2}{n-2}-\sum_{k=3}^{m-1}\frac{n-s}{(n-k)(n-k+1)}\right)$

For this to occur, we must have $\{1,\ldots,m-1\}\subseteq S, S-\{i\}$, with $m<|S|<n-1$. This is only possible for $i\geq m$. Then we want $S=\{1,\ldots,m-1,i\}\cup R$, where $R\subseteq\{m,\cdots,i-1,i+1,\cdots,n\}$ and $|R|=s-m$. There are $\binom{n-m}{s-m}$ such coalitions for each cardinality. Thus, the Shapley value contribution is

$$\sum_{s=m}^{n-2}\binom{n-m}{s-m}\frac{(s-1)!(n-s)!}{n!}\left[\left(\frac{s-1}{n-2}-\sum_{k=3}^{m-1}\frac{n-s-1}{(n-k)(n-k+1)}\right)-\left(\frac{s-2}{n-2}-\sum_{k=3}^{m-1}\frac{n-s}{(n-k)(n-k+1)}\right)\right]$$

From subcases 1-4, we have

$$\varphi_{i<m}(\hat{c})=\frac{1}{n}(1-0)+(n-m+1)\frac{1}{n(n-1)}(1-0)+\sum_{s=m-1}^{n-2}\binom{n-m+1}{s-m+1}\frac{(s-1)!(n-s)!}{n!}\left[\left(\frac{s-1}{n-2}-\sum_{k=3}^{m-1}\frac{n-s-1}{(n-k)(n-k+1)}\right)-0\right]$$

$$=\frac{1}{n}+\frac{n-m+1}{n(n-1)}+\sum_{s=m-1}^{n-2}\frac{(s-1)!}{(s-m+1)!}\frac{(n-m+1)!}{n!}\left[\frac{s-1}{n-2}-\frac{(n-s-1)(m-3)}{(n-2)(n-m+1)}\right]$$

$$= \frac{1}{n} + \frac{n-m+1}{n(n-1)} + \frac{(n-m)!}{n!} \sum_{s=m-1}^{n-2} \frac{(s-m+2)(s-1)!}{(s-m+1)!}$$

(by lemma 1)
$$= \frac{n(m-1)+m}{mn(m-1)}$$

$$\varphi_{i \geq m}(\hat{c}) = (n-m)\frac{1}{n(n-1)}\left(\left(1 - \left(\frac{n-3}{n-2} - \sum_{k=3}^{m-1} \frac{1}{(n-k)(n-k+1)}\right)\right) + \sum_{s=m}^{n-2} \binom{n-m}{s-m} \frac{(s-1)!(n-s)!}{n!}\left[\left(\frac{s-1}{n-2} - \sum_{k=3}^{m-1} \frac{n-s-1}{(n-k)(n-k+1)}\right) - \left(\frac{s-2}{n-2} - \sum_{k=3}^{m-1} \frac{n-s}{(n-k)(n-k+1)}\right)\right]\right.$$

$$= \frac{n-m}{n(n-1)(n-m+1)} + \sum_{s=m}^{n-2} \frac{(n-m)!}{(s-m)!(n-s)!} \frac{(s-1)!(n-s)!}{n!}\left[\frac{1}{n-m+1}\right]$$

$$= \frac{n-m}{n(n-1)(n-m+1)} + \frac{(n-m)!}{(n-m+1)\,n!} \sum_{s=m}^{n-2} \frac{(s-1)!}{(s-m)!}$$

$$= \frac{n-m}{n(n-1)(n-m+1)} + \frac{(n-m)!}{(n-m+1)n!} \frac{(n-2)!}{m(n-m-2)!}$$

$$= \frac{n-m}{mn(n-m+1)}$$

se 4: Let $a_0 = a_n = \ldots = a_2 = 1$, $a_1 = 0$.

Again we only need to compute the Shapley value of $\hat{c}$ for coalitions where the player's marginal contribution is nonzero.

By Lemma 1 of Appendix A, Theorem 1 can be simplified to: $\hat{c}(S) = \omega(S)$ if $|S| \in J$,

$$\hat{c}(S) = \frac{s-1}{n-2} a_{i(S)} \text{ if } |S| \notin J.$$

There are then four possible nonzero marginal contributions: $1-0$, $\frac{s-1}{n-2} - 0$, $1 - \frac{s-2}{n-2}$, and $\frac{s-1}{n-2} - \frac{s-2}{n-2}$.

Subcase 1: Player's marginal contribution equals $1-0$.

There are four ways in which this can occur:

① $S = N$, $S - \{i\} \not\ni 1$.

This is only possible for $i=1$. In this case, the Shapley value contribution is $\frac{1}{n}(1-0)$.

② $S=N$, $|S-\{i\}|=1$.

Since $n\geq 4$, this case is impossible.

③ $S=N-\{j\}$, $j\neq 1$, $S-\{i\}\not\ni 1$.

This is only possible for $i=1$. Then there are $n-1$ such coalitions S that contain 1, and so the Shapley value contribution is $(n-1)\frac{1}{n(n-1)}(1-0)$.

④ $S=N-\{j\}$, $j\neq 1$, $|S-\{i\}|=1$.

Since $n\geq 4$, this case is impossible.

Subcase 2: Player's marginal contribution equals $1-\frac{S-2}{n-2}$.

To obtain $\hat{c}(S)=1$, we can have $S=N$ or $S=N-\{j\}$, $j\neq 1$.

But in order to obtain $\hat{c}(S-\{i\})=\frac{S-2}{n-2}$, we must have $1<|S-\{i\}|<n-1$ with $1\in S-\{i\}$.

$|S|$ must then equal $n-1$, and therefore $S=N-\{j\}$, $j\neq 1$.

For $i=1$, these conditions lead to a contradiction. For $i\neq 1$, there are $n-2$ such coalitions that contain i. The Shapley value contribution is then

$$(n-2)\frac{1}{n(n-1)}\left(1-\frac{n-3}{n-2}\right).$$

Subcase 3: Player's marginal contribution equals $\frac{S-1}{n-2}-0$.

For this to occur, we must have $S\ni 1$ with $1<|S|<n-1$ and either $S-\{i\}\not\ni 1$ or $|S-\{i\}|=1$.

① $S-\{i\}\not\ni 1$.

This is only possible for $i=1$. Then we want $S=\{1\}\cup R$, where $R\subset\{2,\ldots,n\}$ and $|R|=S-1$. There are $\binom{n-1}{S-1}$ such coalitions for each cardinality of S. Thus, the Shapley value contribution is $\sum_{S=2}^{n-2}\binom{n-1}{S-1}\frac{(S-1)!(n-S)!}{n!}\left[\frac{S-1}{n-2}-0\right]$.

② $|S-\{i\}|=1$

This implies $|S|=2$. If $i=1$, then this case is the same as ①.

If $i\neq 1$, then there exists only one coalition S such that $S\ni 1,i$ and $|S|=2$.

The Shapley value contribution is then $\frac{1}{n(n-1)}\left[\frac{1}{n-2}-0\right]$.

Subcase 4: Player's marginal contribution equals $\frac{s-1}{n-2} - \frac{s-2}{n-2}$.

For this to occur, we must have $1 \in S, S-\{i\}$, with $1<|S|<n-1$.

This is only possible for $i \neq 1$. Then we want $S=\{1,i\} \cup R$, where $R \subseteq \{2,\dots,i-1,i+1,\dots,n\}$ and $|R| = s-2$. There are $\binom{n-2}{s-2}$ such coalitions for each cardinality. Thus, the Shapley value

ntribution is $\displaystyle\sum_{s=3}^{n-2} \binom{n-2}{s-2} \frac{(s-1)!(n-s)!}{n!} \left[\frac{s-1}{n-2} - \frac{s-2}{n-2}\right]$.

rom subcases 1-4, we have

$$\varphi_1(\hat{c}) = \frac{1}{n}(1-0) + (n-1)\frac{1}{n(n-1)}(1-0) + \sum_{s=2}^{n-2} \binom{n-1}{s-1} \frac{(s-1)!(n-s)!}{n!}\left[\frac{s-1}{n-2} - 0\right]$$

$$= \frac{1}{n} + \frac{1}{n} + \sum_{s=2}^{n-2} \frac{s-1}{n(n-2)}$$

$$= \frac{1}{n(n-2)} \sum_{s=2}^{n-2} (s-1) + \frac{2}{n}$$

$$= \frac{1}{n(n-2)} \sum_{s=1}^{n-3} s + \frac{2}{n}$$

$$= \frac{(n-3)(n-2)}{2n(n-2)} + \frac{2}{n}$$

$$= \frac{n+1}{2n}$$

$$\varphi_i(\hat{c}) = (n-2)\frac{1}{n(n-1)}\left(1 - \frac{n-3}{n-2}\right) + \sum_{s=3}^{n-2} \binom{n-2}{s-2} \frac{(s-1)!(n-s)!}{n!}\left[\frac{s-1}{n-2} - \frac{s-2}{n-2}\right] + \frac{1}{n(n-1)}\left[\frac{1}{n-2} - 0\right]$$

$$= \frac{1}{n(n-1)} + \frac{1}{n(n-1)(n-2)} \sum_{s=3}^{n-2} (s-1) + \frac{1}{n(n-2)(n-1)}$$

$$= \frac{1}{n(n-1)} + \frac{n-4}{2n(n-2)} + \frac{1}{n(n-2)(n-1)}$$

$$= \frac{1}{2n}$$

se 5: Let $a_0 = a_n = a_{n-1} = \dots = a_1 = 1$.

Following case 1, symmetry and efficiency imply $\varphi_i(\hat{c}) = \frac{1}{n}$.

ve now find θ for the above basis.

ase 1: Let $a_0=1, a_n=\ldots=a_1=0$

Then $\theta_1=\frac{1}{n}$

$\theta_i=\frac{1}{n}$.

ase 2: Let $a_0=a_n=1$, $a_{n-1}=\ldots=a_1=0$.

Then $\theta_1=\frac{(n-1)n+(n-1+1)}{(n-1)(n-1+1)n}=\frac{1}{n-1}$

$$\theta_{1<i<n}=\frac{(n-1)n+(n-1+1)}{(n-1)(n-1+1)n}=\frac{1}{n-1}$$

$$\theta_n=\frac{n-(n-1+1)}{n(n-(n-1))(n-1+1)}=0$$

ase 3: Let $a_0=a_n=\ldots=a_m=1, a_{m-1}=\ldots=a_1=0$, $(n>m>2)$

Then $\theta_1=\frac{(m-1)n+(m-1+1)}{(m-1)(m-1+1)n}=\frac{n(m-1)+m}{nm(m-1)}$

$$\theta_{1<i<m}=\frac{n(m-1)+m}{nm(m-1)}$$

$$\theta_{i\geq m}=\frac{n-(m-1+1)}{n(n-(m-1))(m-1+1)}=\frac{n-m}{mn(n-m+1)}$$

ase 4: Let $a_0=a_n=\ldots=a_2=1$, $a_1=0$.

Then $\theta_1=\frac{n+1}{2n}$

$\theta_i=\frac{1}{2n}$

ase 5: Let $a_0=a_n=\ldots=a_1=1$

Then $\theta_1=\frac{1}{n}$

$\theta_i=\frac{1}{n}$.

It follows that $\theta_i=\varphi_i(\hat{c})$ on a basis for the a_j's, and therefore we have proven the result.

$\square$

ppendix C

We will need the following Lemma in the proof of Theorem 4:

mma 1

$$\sum_{s=m-1}^{n-2} \binom{n-m+1}{s-m+1} \frac{(s-1)!(n-s)!}{n!} \left(\frac{1}{2}-0\right) = \frac{(n-m)(n-m+1)}{2n(n-1)(m-1)}$$

oof: Through algebraic manipulation, the above statement reduces to:

$$\sum_{s=m-1}^{n-2} \frac{(s-1)!}{(s-m+1)!} = \frac{(n-2)!}{(m-1)(n-m-1)!}$$

We use induction on n.

Base step: Let $n=2$.

$$\sum_{s=m-1}^{0} \frac{(s-1)!}{(s-m+1)!} = 0 = \frac{(2-2)!}{(m-1)(n-m-1)!}$$

nductive hypothesis: Assume $\sum_{s=m-1}^{n-3} \frac{(s-1)!}{(s-m+1)!} = \frac{(n-3)!}{(m-1)(n-m-2)!}$

We show that $\sum_{s=m-1}^{n-2} \frac{(s-1)!}{(s-m+1)!} = \frac{(n-2)!}{(m-1)(n-m-1)!}$

$$\sum_{s=m-1}^{n-2} \frac{(s-1)!}{(s-m+1)!} = \sum_{s=m-1}^{n-3} \frac{(s-1)!}{(s-m+1)!} + \frac{(n-3)!}{(n-m-1)!}$$

(by inductive hypothesis)

$$= \frac{(n-3)!}{(m-1)(n-m-2)!} + \frac{(n-3)!}{(n-m-1)!}$$

$$= \frac{(n-3)!\,(n-m-1+m-1)}{(m-1)(n-m-1)!}$$

$$= \frac{(n-3)!\,(n-2)}{(m-1)(n-m-1)!}$$

$$= \frac{(n-2)!}{(m-1)(n-m-1)!}$$

□

Proof of Theorem 4:

Note that X is linear in a_j while the Shapley value is linear in X. Therefore, Ψ is linear in a_j. It follows that Ψ can be uniquely determined on a basis for the a_j's.

We first find $\varphi(X)$ for the following basis:

$a_0 = 1,\ a_n = \dots = a_1 = 0$

$a_0 = a_n = 1,\ a_{n-1} = \dots = a_1 = 0$

$a_0 = a_n = \dots = a_m = 1,\ a_{m-1} = \dots = a_1 = 0 \quad ,(n > m > 2)$

$a_0 = a_n = \dots = a_2 = 1,\ a_1 = 0$

$a_0 = a_n = \dots = a_1 = 1$

Case 1: Let $a_0 = 1,\ a_n = \dots = a_1 = 0$

Then it follows from symmetry that each player should receive the same payoff.

By efficiency, we have $\varphi_i(X) = \frac{1}{n}$.

Case 2: Let $a_0 = a_n = 1,\ a_{n-1} = \dots = a_1 = 0$.

For $|S| < n-1$, $i(S) \leq n-1 \Rightarrow a_{i(S)} = 0$. Therefore, $X(S) = \begin{cases} 1, & S = N,\ N - \{n\} \\ 0, & \text{otherwise} \end{cases}$

It then follows that $\varphi_{i<n}(X) = \frac{1}{n}(1-0) + \frac{1}{n(n-1)}(1-0) = \frac{1}{n-1}$

$$\varphi_n(X) = \frac{1}{n}(1-1) = 0$$

Case 3: Let $a_0 = a_n = \dots = a_m = 1,\ a_{m-1} = \dots = a_1 = 0,\ (n > m > 2)$

We only need to compute the Shapley value of X for coalitions where the player's marginal contribution is nonzero. There are three possibilities: the marginal contribution can equal $1-0$, $1-\frac{1}{2}$, or $\frac{1}{2}-0$.

Subcase 1: Players marginal contribution equals $1-0$.

There are four ways in which this can occur:

① $S = N$, $S - \{i\} \not\ni K$, where $K < m$.

This is only possible for $i < m$. The Shapley value contribution is then $\frac{1}{n}(1-0)$.

② $S=N$, $|S-\{i\}|=1$.

Since $n \geq 4$, this case is impossible.

③ $S=N-\{j\}$, $j \geq m$, $S-\{i\} \not\ni k$, where $k<m$

This can only occur when $i<m$. There are $n-m+1$ such coalitions S that contain i, and so the Shapley value contribution is $(n-m+1)\frac{1}{n(n-1)}(1-0)$.

④ $S=N-\{j\}$, $j \geq m$, $|S-\{i\}|=1$.

Since $n \geq 4$, this case is impossible.

Subcase 2: Player's marginal contribution equals $1-\frac{1}{2}$.

To obtain $X(S)=1$, we can have $S=N$ or $S=N-\{j\}$, $j \geq m$.

But in order to obtain $X(S-\{i\})=\frac{1}{2}$, we must have $1<|S-\{i\}|<n-1$ with $\{1,\dots,m-1\} \subset S-\{i\}$. $|S|$ must then equal $n-1$, and therefore $S=N-\{j\}$, $j \geq m$.

For $i<m$, these conditions lead to a contradiction. For $i \geq m$, there are $n-m$ such coalitions that contain i, and so the Shapley value contribution is $(n-m)\frac{1}{n(n-1)}(1-\frac{1}{2})$.

Subcase 3: Player's marginal contribution equals $\frac{1}{2}-0$.

For this to occur, we must have $\{1,\dots,m-1\} \subseteq S$, with $m-1 \leq |S| < n-1$ and either $\exists\, k<m$ such that $S-\{i\} \not\ni k$ or $|S-\{i\}|=1$.

① $\exists\, k<m$ such that $S-\{i\} \not\ni k$

This is only possible for $i<m$. Then we want $S=\{1,2,\dots,m-1\} \cup R$, where $R \subset \{m, m+1, \dots, n\}$ and $|R|=s-m+1$. There are $\binom{n-m+1}{s-m+1}$ such coalitions for each cardinality. Thus, the Shapley value contribution is

$$\sum_{s=m-1}^{n-2} \binom{n-m+1}{s-m+1} \frac{(s-1)!\,(n-s)!}{n!} \left(\frac{1}{2}-0\right).$$

② $|S-\{i\}|=1$.

This implies $|S|=2$. If $i<m$, then this case is the same as ①.

If $i \geq m$, then this case is impossible, since $S \supseteq \{1, \dots, m-1, i\}$ and therefore $|S|>2$.

From subcases 1-3, we have

$$\varphi_{i<m}(X) = \frac{1}{n}(1-0) + (n-m+1)\frac{1}{n(n-1)}(1-0) + \sum_{S=m-1}^{n-2} \binom{n-m+1}{S-m+1} \frac{(S-1)!(n-S)!}{n!} \left(\tfrac{1}{2}-0\right)$$

$$\overset{\text{(by Lemma 1)}}{=} \frac{1}{n} + \frac{n-m+1}{n(n-1)} + \frac{(n-m)(n-m+1)}{2n(n-1)(m-1)}$$

$$= \frac{2n(n-1)-(n-m)(n-m+1)}{2n(m-1)(n-1)}$$

$$\varphi_{i\geq m} = (n-m)\frac{1}{n(n-1)}\left(1-\tfrac{1}{2}\right) = \frac{n-m}{2n(n-1)}$$

ase 4: Let $a_0 = a_n = \ldots = a_2 = 1$, $a_1 = 0$.

Again we only need to compute the Shapley value of X for coalitions where the player's marginal contribution is nonzero. The three possibilities are: $1-0$, $1-\frac{1}{2}$, and $\frac{1}{2}-0$.

Subcase 1: Player's marginal contribution equals $1-0$.

There are four ways in which this can occur:

① $S=N$, $S-\{i\} \not\ni 1$.

This is only possible for $i=1$. In this case, the Shapley value contribution is $\frac{1}{n}(1-0)$.

② $S=N$, $|S-\{i\}|=1$.

Since $n\geq 4$, this case is impossible

③ $S=N-\{j\}$, $j\neq 1$, $S-\{i\} \not\ni 1$

This is only possible for $i=1$. Then there are $n-1$ such coalitions S that contain 1, and so the Shapley value contribution is $(n-1)\frac{1}{n(n-1)}(1-0)$.

④ $S=N-\{j\}$, $j\neq 1$, $|S-\{i\}|=1$

Since $n\geq 4$, this case is impossible.

Subcase 2: Player's marginal contribution equals $1-\frac{1}{2}$.

To obtain $X(S)=1$, we can have $S=N$ or $S=N-\{j\}$, $j\neq 1$.

But in order to obtain $X(S-\{i\})=\frac{1}{2}$, we must have $1<|S-\{i\}|<n-1$, with $1\in S-\{i\}$.

$|S|$ must then equal $n-1$, and therefore $S=N-\{j\}$, $j\neq 1$.

For $i=1$, these conditions lead to a contradiction. For $i \neq 1$, there are $n-2$ such coalitions that contain i. The Shapley value contribution is then $(n-2)\frac{1}{n(n-1)}(1-\frac{1}{2})$.

Subcase 3: Player's marginal contribution equals $\frac{1}{2}-0$.

For this to occur, we must have $S \ni 1$, with $1<|S|<n-1$ and either $S-\{i\} \not\ni 1$ or $|S-\{i\}|=1$.

① $S-\{i\} \not\ni 1$

This is only possible for $i=1$. Then we want $S=\{1\} \cup R$, where $R \subseteq \{2,\ldots,n\}$ and $|R|=s-1$.

There are $\binom{n-1}{s-1}$ such coalitions for each cardinality of S. Thus, the Shapley value contribution is $\sum_{s=2}^{n-2}\binom{n-1}{s-1}\frac{(s-1)!(n-s)!}{n!}(\frac{1}{2}-0)$.

② $|S-\{i\}|=1$

This implies $|S|=2$. If $i=1$, then this case is the same as ①.

If $i \neq 1$, then there exists only one coalition S such that $S \ni 1, i$, and $|S|=2$. The Shapley value contribution is $\frac{1}{n(n-1)}(\frac{1}{2}-0)$.

om subcases 1-3, we have

$$\varphi_1(X) = \frac{1}{n}(1-0) + (n-1)\frac{1}{n(n-1)}(1-0) + \sum_{s=2}^{n-2}\binom{n-1}{s-1}\frac{(s-1)!(n-s)!}{n!}(\tfrac{1}{2}-0)$$

$$= \frac{1}{n} + \frac{1}{n} + \sum_{s=2}^{n-2}\frac{1}{2n}$$

$$= \frac{2}{n} + \frac{n-3}{2n}$$

$$= \frac{n+1}{2n}$$

$$\varphi_{i>1}(X) = (n-2)\frac{1}{n(n-1)}(1-\tfrac{1}{2}) + \frac{1}{n(n-1)}(\tfrac{1}{2}-0)$$

$$= \frac{1}{2n}$$

se 5: Let $a_0=a_n=a_{n-1}=\ldots=a_1=1$

Following case 1, symmetry and efficiency imply $\varphi_{i>1}(X)=\frac{1}{n}$.

We now find Ψ for the above basis.

Case 1: Let $a_0=1$, $a_n=\dots=a_1=0$

Then $\Psi_1=\frac{1}{n}$

$\Psi_{i>1}=\frac{1}{n}$.

Case 2: Let $a_0=a_n=1$, $a_{n-1}=\dots=a_1=0$

Then $\Psi_1=\frac{1}{n-1}$

$\Psi_{1<i<n}=\frac{1}{n-1}$

$\Psi_n=0$

Case 3: Let $a_0=a_n=\dots=a_m=1$, $a_{m-1}=\dots=a_1=0$, $(n>m>2)$

Then $\Psi_1=\frac{2n(n-1)-(n-m)(n-m+1)}{2n(m-1)(n-1)}$

$\Psi_{1<i<m}=\frac{2n(n-1)-(n-m)(n-m+1)}{2n(m-1)(n-1)}$

$\Psi_{i\geq m}=\frac{n-m}{2n(n-1)}$

Case 4: Let $a_0=a_n=\dots=a_2=1$, $a_1=0$

Then $\Psi_1=\frac{n+1}{2n}$

$\Psi_{i>1}=\frac{1}{2n}$

Case 5: Let $a_0=a_n=a_{n-1}=\dots=a_1=1$

Then $\Psi_1=\frac{1}{n}$

$\Psi_{i>1}=\frac{1}{n}$.

It follows that $\Psi_i=\Phi_i(X)$ on a basis for the a_j's, and therefore we have proven the result.

$\square$