Characterization of the Shapley-Shubik and Banzhaf Power Indices with Transfer and Fairness Axioms.

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Characterization of Power Indices

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The Shapley-Shubik and Banzhaf power indices can be characterized on simple monotonic games with Efficiency and two classes of axioms. The first class, the Transfer axioms, partially dictates the structure of the indices and includes the familiar axioms used by Dubey in his characterization (1975). The second class, the Fairness axioms, pertains to the notion of 'fair play.' It includes notions such as Symmetry. We also show that both power indices can be characterized on superadditive simple games by axioms weaker than those previously examined.

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List of Symbols:

$\cap$	Intersection
$\cup$	Union
$\supset$	Strict superset of
$\supseteq$	Superset of
$\subset$	Strict subset of
$\subseteq$	Subset of
$\forall$	For all
$\ni$	Such that
$\varnothing$	Empty set
$\in$	Member of
$\notin$	Not a member of
Γ	Capital Gamma
ψ	Psi
Σ	Capital Sigma
φ	Phi
η	Nu
$\vert$	Bar
β	Beta
π	Pi
$\rightarrow$	Right Arrow
$\wedge$	Logical And
$\vee$	Logical Or

§0: Introduction.

The Shapley-Shubik and Banzhaf power indices are quantitative measures of the influence of a given player in a simple cooperative game. These indices have been previously characterized by Dubey with the familiar axioms of transfer, symmetry, dummy and efficiency. We present generalizations of these axioms, which fall into three categories. The first such is the Transfer axioms, which govern the behavior of a power index for a fixed player across different games. The second is concerned with the behavior of a power index for a fixed game across different players. The Equal Treatment axiom is a member of this category. The other Fairness axioms which we propose incorporate aspects of both these classes. The third grouping, to which Efficiency and Efficiency′ belong, specifies the overall power across players in a fixed game. We also introduce a set of axioms weaker than that used by Dubey, and characterize both power indices with it.

§1: Preliminaries.

Let $N = \{1, \ldots, n\}$ be the set of *players*. A *coalition* of players is $S \subseteq N$. Denote by n and s the cardinality of N and S respectively.

A *game* is a real-valued set function v defined on all coalitions and satisfying $v(\emptyset) = 0$. A *simple game* is a game such that $v(S) = 1$ or 0 for all S, and $v(N) = 1$. A coalition S is *winning* whenever $v(S) = 1$ and *losing* otherwise. A game is *monotonic* if $v(S) \geq v(S')$ whenever $S \supseteq S'$. A game is *superadditive* if $v(S \cup T) \geq v(S) + v(T)$ for $S \cap T = \emptyset$. A game is *proper* if it is both monotonic and superadditive. In the class of simple games, superadditivity requires that disjoint coalitions cannot both be winning. Let Γ and C denote the class of games, and the class of simple monotonic games respectively.

A *value* is a function ψ that associates with every game $v \in \Gamma$ a real n-vector $\psi(v) \equiv (\psi_i(v))$, $i \in N$. A *power index* is a value restricted to C.

For simple games, if $v(S) = 1$ and $v(S - i) = 0$, then we say that player i is *critical* with respect to S in v, and S is a *pivot* for i in v. Let $P_i(v)$ be the collection of pivots for i in v. A coalition is a *minimal winning coalition* (MWC) whenever all its members are critical. A monotonic simple game can be specified by a listing of its MWCs: $v(S) = 1$ if $T \subseteq S$ for some MWC T; otherwise, $v(S) = 0$.

A *unanimity game* u_T is a game such that $u_T(S) = 1$ if and only if $S \supseteq T$; i.e. a simple monotonic game in which T is the only MWC.

§2: The Shapley-Shubik and Banzhaf Power Indices.

The Shapley value on Γ is defined by

$$\varphi_i(v) = \sum \frac{(s-1)!(n-s)!}{n!}(v(S) - v(S - i))$$

where the sum is taken over all coalitions containing player i.

The Shapley-Shubik power index on C is defined by

$$\varphi_i(v) = \sum_{S \in P_i(v)} \frac{(s-1)!(n-s)!}{n!}.$$

Denote by $\eta_i(v)$ the number of coalitions to which i is critical, i.e. $\eta_i(v) = |P_i(v)|$. These are known as the "raw Banzhaf indices." Let $\eta'(v) = \sum_{i \in N} |P_i(v)|$.

The Banzhaf power index is defined by

$$\beta i(v) = \frac{\eta_i(v)}{\eta'(v)}.$$

The absolute Banzhaf power index is defined by

$$\beta i(v) = \frac{\eta_i(v)}{2^{n-1}}.$$

The Shapley-Shubik power index can be thought of as the probability that a player is critical, assuming that all permutations are equally likely to occur. The absolute Banzhaf power index is a similar probability measure, but assumes instead that all combinations of players are equally likely. The Banzhaf power index is the normalization of the absolute index.

We introduce a number of value axioms in this paper. The format of our definitions is, "The value ψ satisfies axiom A (on class G of games) if condition C holds (for all games in G)." In order to conserve space, the references to the class G in parentheses has been omitted from each of our definitions. The theorems will explicitly state the class of games under consideration.

Shapley, in 1953, proposed the following four axioms:

Efficiency (Eff): A value ψ is efficient if $\sum_{i \in N} \psi_i(v) = v(N)$.

Note that on C, ψ is efficient if $\sum_{i \in N} \psi_i(v) = 1$.

For the set of players N, define $\pi: N \to N$ be a permutation of N. For all $S \subseteq N$, let $\pi v(S) = v(\pi(S))$.

Symmetry (Sym): The value ψ is symmetric if $\forall\, i \in N$ and for all permutations π of N, $\psi_{\pi(i)}(v) = \psi_i(\pi v)$.

The player i is a dummy in the game v if $v(S \cup i) - v(S) = 0$ for all $S \subseteq N$.

Dummy (Du): The value ψ satisfies the dummy axiom if $\psi_i(v) = 0$ whenever player i is a dummy in v.

Given two games $v, w \in \Gamma$, let $(v + w)(S) = v(S) + w(S)$ for all $S \subseteq N$. It is easy to show that $v + w \in \Gamma$.

Additivity (Add): The value ψ is additive if $\psi(v + w) = \psi(v) + \psi(w)$.

Theorem (Shapley, 1953): The Shapley value is the unique value satisfying Eff, Sym, Du and Add on the class of all (superadditive) games.

In fact, the Shapley value can be characterized through a weaker axiom, Equal Treatment (ET), in place of Sym.

Equal Treatment (ET): The value ψ satisfies the equal treatment axiom if $\psi_i(v) = \psi_j(v)$ for all $S \subseteq N - \{i,j\}$.

Eff, Sym, Du and Add do not uniquely specify the Shapley value on C since Add is not a restriction on simple games. Indeed, for v,w $\in$ C, the game $v + w \notin C$, as $(v + w)(N) = 2$. In order to characterize the value on C, Dubey (1975) replaced Add with a transfer axiom.

Dubey Transfer Axiom (DTr): The power index ψ satisfies the Dubey transfer axiom if $\psi(v \wedge w) + \psi(v \vee w) = \psi(v) + \psi(w)$, where $v \wedge w$ and $v \vee w$ denote the games given by

$$(v \wedge w)(S) \begin{cases} 1 \text{ if } v(S) = 1 \text{ and } w(S) = 1 \\ 0 \quad \text{otherwise} \end{cases}$$

$$(v \vee w)(S) \begin{cases} 1 \text{ if } v(S) = 1 \text{ or } w(S) = 1 \\ 0 \quad \text{otherwise} \end{cases}$$

Theorem (Dubey, 1975): The Shapley-Shubik power index is the unique value satisfying Eff, Sym, Du and DT on the class of monotonic simple games.

Dubey (1979) characterized the Banzhaf index β for simple monotonic games with the axioms Sym, Du, DT and a variation of Eff. For β, a power index satisfies Dubey's modified Efficiency (Eff′) if

$$\sum_{i \in N} \psi_i(v) = \eta'(v).$$

Theorem (Dubey, 1979): The raw Banzhaf indices is the unique value satisfying Eff′, Sym, Du, and DT on the class of monotonic simple games.

§3: Transfer Axioms.

We now propose axioms which will generalize and strengthen these characterizations of the Shapley-Shubik and the Banzhaf indices. The first set determines the structure of the index.

Unanimity game Transfer (UTr): The power index ψ satisfies the unanimity transfer axiom if for all $T \subseteq N$ such that $T \neq \emptyset$, $\psi_i(v) = \psi_i(v \wedge u_T) + \psi_i(v \vee u_T)$, $\forall\, i \notin T$.

UTr is formulated with respect to a unanimity game applied in conjunction with Dubey's Transfer axiom, and so is roughly analogous to the Coalitional Strategic Equivalence axiom proposed by Chun (1989) in his characterization of the Shapley value. It specifies the effect of an unanimity game u_T on the players outside the coalition T. By interpreting $v \wedge u_T$ and $v \vee u_T$ as games in which, respectively, coalition T has acquired the power to veto motions and the power to pass motions, we can see that UTr states that the sum of the power lost by a player $i \notin T$ in each new situation is equal to his total power in the original game v. Note that if v is a superadditive game, $v \vee u_T$ need not be superadditive. UTr holds on superadditive games if the given equality is true for all superadditive games v for which $v \vee u_T$ is also superadditive.

Split Pivot Transfer (STr): The power index ψ satisfies the split pivot transfer axiom if $\psi_i(w) = \psi_i(v) + \psi_i(u)$ when the games u and v split player i's pivots in the game w, i.e. $P_i(w) = P_i(v) \cup P_i(u)$ and $P_i(v) \cap P_i(u) = \emptyset$.

STr requires that if the pivots of player i are completely divided between two other games, then the worth of i in the original game is equal to the sum of the worth in the other games. This is in a sense similar to Shapley's original Additivity axiom. It formalizes, in a strong form, the notion that a player's voting power is the sum of the power available in each possible situation.

TransferSum Axiom: The power index ψ satisfies the TransferSum Axiom if there exists a function $f : 2^N \times \{1,2, . . . ,n\} \to \Re$, where 2^N is the space of subsets of N, such that $\psi_i(v) = \sum_{S \in P_i(v)} f(S,i)$.

Note that the definition of f(S,i) only matters for $i \in S$.

Weber 1978 demonstrated that if $\psi_i(v)$ is a value on simple games satisfying DTr, Dummy, and Monotonicity (if a game is monotonic, then $\psi_i(v) \geq 0$), then there is a collection of constants $\{p_T : T \subseteq N\text{-}i\}$ satisfying $\sum_{T \subseteq N\text{-}i} p_T = 1$ and $p_T \geq 0$ for all $T \neq \emptyset$, such that for every game $v \in C$, $\sum_{T \subseteq N\text{-}i} p_T[v(T \cup i) - v(T)]$.

Our function f(S,i) is a probability distribution on the coalitions containing player i when Weber's conditions are applied. The power index can be thought of as the probability that the player is critical.

Bolger 1980 proposed characterizations of values of the form $\psi_i(v) = \sum_{S \in P_i(v)} f(S,v)$. Power indices of this form include the Shapley-Shubik, Banzhaf and Deegan-Packel. Our axiom states that such a summation is a basic property, though we are considering only functions f that do not depend on the game v.

It is possible to prove (see steps 1a, 2a, and 3a in the proof of Theorem 1 below) directly from the definitions that (1) TransferSum implies STr, DTr, and Du, (2) STr implies UTr, and (3) DTr and Du imply UTr. Our first theorem says that all four transfer axioms are equivalent on monotonic simple games. However, UTr is strictly weaker than the others on proper (i.e., superadditive and monotonic) simple games.

Theorem 1: The following *transfer axioms* are equivalent on monotonic simple games. The first three axioms are equivalent and imply the fourth on proper simple games.

1) TransferSum Axiom
2) Split Pivot Transfer Axiom
3) Dubey's Transfer Axiom and Dummy
4) Unanimity Transfer Axiom.

The proof of the equivalence on monotonic simple games is divided into three steps.

Step 1a: The TransferSum axiom implies STr.

Proof: We can simply separate the sum by the coalitions split by games u and v. Obviously, this implies $\psi_i(w) = \psi_i(v) + \psi_i(u)$.

Step 1b: STr implies the TransferSum axiom.

Proof: Define v^T to be the game with MWCs $T \cup \{j\}$ for all $j \notin T$. Define $f(S,i) = \psi_i(v^{S-i})$. Clearly, player i has exactly one pivot in v^{S-i}, namely S, so TransferSum holds on all such games. We will show that TransferSum holds on all monotonic simple games through an induction on the number of i's pivots.

If player i is a dummy in w, then w and w vacuously split i's pivots. STr implies that $\psi_i(w) = \psi_i(w) + \psi_i(w)$, hence $\psi_i(w) = 0$. So, TransferSum holds on all games having zero pivots for player i.

Assume TransferSum holds for games where i has m-1 or fewer pivots. Let w be a game where i has m > 0 pivots. There must exist a MWC T containing i. Let u have the same winning coalitions as w except for T. So, player i has the same pivots in u as in w except for T. Let $v = v^{T-i}$. So, T is the only pivot for player i in v. Hence, u and v split the pivots of w and i has m -1 pivots in u. By the inductive assumption, the definition of f, and STr, we may write $\psi_i(w)$ in summation form.

<u>Step 2a:</u> The TransferSum axiom implies Du and DTr.

<u>Proof:</u> If player i is a dummy, then the sum is vacuous in TransferSum. So, Du follows. Consider now two games v and w. If S is a pivot for i in both v and w, then S is a pivot for i in both $v \wedge w$ and $v \vee w$. If S is a pivot for i in exactly one of v or w, then S is a pivot for i in exactly one of $v \wedge w$ or $v \vee w$. Finally, if S is a pivot for in in neither v or w, then S is a pivot for i in neither one of $v \wedge w$ or $v \vee w$. Hence, it is possible to rewrite

$$\sum_{S \in P_i(v)} f(S,i) + \sum_{S \in P_i(w)} f(S,i)$$

as

$$\sum_{S \in P_i(v \wedge w)} f(S,i) + \sum_{S \in P_i(v \vee w)} f(S,i)$$

which shows that DTr holds.

<u>Step 2b:</u> Du and DTr imply the TransferSum axiom.

<u>Proof:</u> We define f(S,i) inductively. Let $f(N,i) = \psi_i(v_N)$, where v_N is the unanimity game on N, and let $f(S,i) = \psi_i(v_S) - \sum f(T,i)$, where the sum is over strict supersets T of S. In this way, we ensure

that TransferSum holds on unanimity games. Clearly, TransferSum also holds on games for which i is a dummy.

We will now consider games in which player i appears in all MWCs, and induct on the number of MWCs. Assume the claim for games of m-1 or fewer MWCs. A game with m MWCs may be written as $v = v_T \vee v_2 \vee \ldots \vee v_m$.

Let $w = v_T$ and $u = v_2 \vee \ldots \vee v_m$. Then by DTr, we have

$$\psi_i(w) + \psi_i(u) = \psi_i(w \wedge u) + \psi_i(v).$$

So $\psi_i(v) = \psi_i(u) + \psi_i(w) - \psi_i(w \wedge u)$. The games u, w and $w \wedge u$ are games with fewer than m MWCs, all of which contain player i. By the induction assumption, $\psi_i(u)$, $\psi_i(w)$ and $\psi_i(w \wedge u)$ can be written in summation form, which yields the desired summation for $\psi_i(v)$ as argued in Step 2a. Consider a game $v = w \vee u$, where w represents MWCs not containing player i, and u the MWCs containing i. By DTr, we have $\psi_i(v) = \psi_i(u) + \psi_i(w) - \psi_i(w \wedge u)$. By Du, $\psi_i(w) = 0$. The games u and $w \wedge u$ are ones that have player i in all MWCs, so they can be expressed in summation form. $P_i(w \wedge u) = P_i(u) - P_i(v)$, so $\psi_i(v)$ can also be expressed in summation form.

<u>Step 3a:</u> Du and DTr imply UTr. STr implies UTr.
<u>Proof:</u> Assume the hypothesis for UTr. By DTr, we have $\psi_i(v) + \psi_i(u_T) = \psi_i(v \wedge u_T) + \psi_i(v \vee u_T)$. By Du, $\psi_i(u_T) = 0$ for all $i \notin T$, as i is a dummy in u_T. Therefore, $\psi_i(v) = \psi_i(v \wedge u_T) + \psi_i(v \vee u_T)$, $\forall i \notin T$.

Assume the hypothesis for UTr again. Clearly, $v \wedge u_T$ and $v \vee u_T$ split the pivots in v of each player $i \notin T$ between supersets and nonsupersets of T, respectively. The conclusion of UTr now follows directly from the conclusion of STr.

Step 3b: UTr implies the TransferSum axiom.

Proof: We first show that UTr implies Du. Let $v = u_T$, the unanimity game on T. Then for $i \notin T$, we have $\psi_i(v) = \psi_i(v \wedge u_T) + \psi_i(v \vee u_T) = 2\psi_i(v)$. Therefore, $\psi_i(v) = 0$ for all $i \notin T$. We now induct on the number of MWCs of v. Assume that UTr implies Du for games with fewer than m MWCs. Consider $v = v_T \vee v_1 \vee \ldots \vee v_{m-1} = v_T \vee v'$, where $v' = v_1 \vee \ldots \vee v_{m-1}$. For i dummy, $\psi_i(v') = \psi_i(v' \wedge v_T) + \psi_i(v' \vee v_T)$ by UTr. The game $v' \vee v_T$ is merely v. The games v' and $v' \wedge v_T$ have fewer than m MWCs, and so the values of i on those games are zero by induction. Therefore, $\psi_i(v) = 0$. Thus, UTr implies Du.

In order to show that UTr implies TransferSum, define f(S,i) inductively as in Step 2b. In this way, TransferSum holds on unanimity games, that is, games with one MWC containing player i and no MWC not containing player i.

Assume that TransferSum holds for games with one MWC containing player i and fewer than $m > 0$ MWC not containing player i. Such a game can be written as $v = v_T \vee v_1 \vee \ldots \vee v_m$, where $i \in T$ and v_k is the unanimity game on the kth MWC not containing player i.

Let $v' = v_T \vee v_1 \vee \ldots \vee v_{m-1}$. By UTr, we have $\psi_i(v') = \psi_i(v) + \psi_i(v' \wedge v_m)$.

Game v' fulfills the induction hypothesis, and $v' \wedge v_m$ either fulfills the induction hypothesis or has i as a dummy. Hence, both games satisfy TransferSum. Since they also split i's pivots in v, it follows that v satisfies TransferSum.

We now induct on the number of MWC containing i. Assume the hypothesis for games with fewer than m > 1 such MWC, and consider the game $v = v_T \vee v_1 \vee \ldots \vee v_{m-1} \vee v'$, where T is the mth MWC containing player i, v_k is the unanimity game on the kth such MWC, and v′ is the game composed of the MWCs not containing i. By UTr,

$$\psi_i(v) = \psi_i(v \wedge v_{T-i}) + \psi_i(v \vee v_{T-i})$$

Both games on the right hand side have fewer than m MWC containing i and therefore fulfill the induction assumption. Since i's pivots of v are split between $v \wedge v_{T-i}$ and $v \vee v_{T-i}$,

$$\psi_i(v) = \sum_{S \in P_i(v)} f(S,i).$$

This completes the proof of the equivalence of the four axioms on the class of monotonic simple games. The proof of the final statement of the theorem involves checking that the above arguments (except for step 3b) still hold (except for one change) on the class of proper simple games. Step 1a, 2a, and 3a follow directly from the definitions. In step 2b, f is defined via unanimity games, which are superadditive, and if v is superadditive in the two induction arguments, then the other games constructed are easily seen to be superadditive. In step 1b, the games v^{S-i} are all superadditive except when S = {i}. We change this step of the proof by defining $f(\{i\},i) = \psi_i(v_{\{i\}}) - \sum f(S,i)$, where the summation is over all strict supersets S of i. In the induction argument of step 1b, if w is superadditive, then u and v are superadditive unless T = {i}. But if T = {i}, then the superadditivity of w implies that w = $v_{\{i\}}$ and TransferSum holds by our definition of f({i},i).

Q.E.D.

The proof of Step 3b holds in the space of proper simple games except for the final induction. Given a proper game v and MWC T, $v \vee v_{T-i}$ may not be proper. For example, v could be a four player game with the MWCs T = {1,2,3} and {1,4}. Then $v \vee v_{T-1}$ has the MWCs {2,3} and {1,4} and is not superadditive. In fact, UTr is strictly weaker than the other transfer axioms on the class of proper games. We will show this by exhibiting power indices satisfying UTr but not the other transfer axioms on 3-player proper simple games. All 3-player proper simple games are listed below.

S	v_1	v_2	v_3	v_4	v_5	v_6	v_7	v_8	v_9	v_{10}	v_{11}
{1,2,3}	1*	1	1*	1*	1	1	1*	1	1	1	1*
{ ,2,3}	0	1	0	0	1	1	0	1	1	1	0
{1, 3}	0	0	1*	0	1*	0	1*	1*	1	0	1*
{1,2}	0	0	0	1*	0	1*	1*	1*	0	1	1*
{ 3}	0	0	0	0	0	0	0	0	1	0	0
{ 2}	0	0	0	0	0	0	0	0	0	1	0
{1}	0	0	0	0	0	0	0	0	0	0	1*
$\psi_1(vi)$	a	0	a+b	a+c	b	c	e+a	e	0	0	a+b +c+d

a = f({1,2,3},1), b = f({1,3},1), c = f({1,2},1), d = f({1},1)

The numbers 1 and 0 in a matrix entry designate winning and losing coalitions, respectively. Coalitions with asterisks next to them are pivots for player 1 in that particular game. The values a,b,c,d represent the function from the TransferSum axiom on a given pivot of player 1. TransferSum would also imply that e = b + c. UTr places no restriction on e because we would be required to construct a nonsuperadditive game. Therefore, by assuming only UTr, we cannot obtain a value for v_8, nor can we obtain one for v_7, as the only superadditive game we would be allowed to construct is v_8.

Thus, on superadditive games, STr, Du and DTr, and the TransferSum axiom are equivalent notions. All of these are stronger notions than UTr, since UTr is a necessary condition of Du and DTr.

§4: Fairness Axioms.

We now consider axioms related to fairness. These include Symmetry and Equal Treatment. We will introduce two additional notions.

Equal Treatment on Unanimity Games (ETUG): The power index ψ satisfies equal treatment on unanimity games if $\psi_i(v_T) = \psi_j(v_S)$ for all $i \in T$, $j \in S$ whenever $|T| = |S|$ for the coalitions T and S.

ETUG requires the players in the MWCs of unanimity games to be treated equally if these MWCs have the same cardinality.

FairSum Axiom: $\exists\, f : \{1,2, \ldots ,n\} \rightarrow \Re \ni \psi_i(v) = \sum_{S \in P_i(v)} f(s)$.

FairSum is the TransferSum axiom when the function f does not differentiate between any particular player i or coalition S. The function depends only on the cardinality of S.

Theorem 2: Suppose the power index ψ satisfies the TransferSum axiom. The following axioms are equivalent on the class of monotonic simple games.

1) FairSum
2) Symmetry
3) Equal Treatment on Unanimity Games (ETUG)
4) Equal Treatment

Call these the *fairness axioms.*

Proof: It follows from the definitions (even without TransferSum) that FairSum implies Sym, and Sym implies both ETUG and ET. Hence, it is sufficient to show that (1) ETUG implies FairSum, and (2) ET implies FairSum. Suppose ψ satisfies TransferSum and f is the function which defines ψ according to TransferSum. In order to show that ψ satisfies FairSum, it is sufficient to show that f(S,i) depends only on the cardinality of S. Clearly, this is true for S = N if ψ satisfies ETUG or ET.

We first finish the proof of ETUG implies FairSum. Suppose that f(S,i) depends only on the cardinality of S whenever the cardinality of S is greater than t. Let T be a coalition of cardinality t. By TransferSum, $f(T,i) = \psi_i(v_T) - \sum_{S \supset T} f(S,i)$. By the induction hypothesis, f(S,i) = f(s) for all $S \supsetneq T$, and so the sum depends only on the cardinality of T. By ETUG, it follows that $\psi_i(v_R) = \psi_j(v_T)$ for all $i \in R$, $j \in T$, and R having cardinality t.

Hence, f(T,i) depends only on the cardinality of T. The proof that ETUG implies FairSum is now complete by induction.

We now consider the proof of ET implies FairSum. It can be shown that f(S,i) depends only on S by way of the same induction as in the last paragraph. The reason a stronger result is not possible is that by ET, it follows only that $\psi_i(v_R) = \psi_j(v_T)$ for all $i \in R$, $j \in T$, and R = T, that is, we cannot let R be any other set than T. Now consider a coalition S and two players $i \in S$ and $j \notin S$. By ET, $f(S) = \psi_i(v^{S-\{i\}}) = \psi_j(v^{S-\{i\}}) = f(S - \{i\} \cup \{j\})$. Successive application of this equality show that f(S) = f(T) for all coalitions S and T with the same cardinality. Thus, ET implies FairSum.

Q.E.D.

The four fairness properties are not equivalent when a power index only satisfies UTr on the class of proper simple games. Consider the 3-player proper simple games introduced earlier and define the power index ψ as follows:

	v_1	v_2	v_3	v_4	v_5	v_6	v_7	v_8	v_9	v_{10}	v_{11}
$\psi_1(v_i)$	a	0	a+b	a+b	b	b	a+e1	e1	0	0	a+2b +d1
$\psi_2(v_i)$	a	a+b	0	a+b	b	a+e2	b	e2	0	a+2b +d2	0
$\psi_3(v_i)$	a	a+b	a+b	0	a+e3	b	b	e3	a+2b +d3	0	0

The power index ψ satisfies ET iff e1 = e2 = e3; ETUG iff d1 = d2 = d3; Sym iff e1 = e2 = e3 and d1 = d2 = d3.

§5: Characterization by Unanimity Transfer Axiom

Note that ET is a necessary condition of Sym. We will now show that the Shapley-Shubik and Banzhaf indices are characterized in the space of superadditive games by ET and UTr, an axiom strictly weaker than DTr used by Dubey (1975).

Theorem 3: The Shapley-Shubik (resp. Banzhaf) power index is the unique power index satisfying UTr, ET and Eff (resp. Eff′) on the class of proper simple games.

Proof: UTr implies Du, and by Eff, ET, and Du, the power index is characterized on unanimity games. We will now induct on the number of MWCs a game has. Assume the characterization for games of m-1 or fewer MWCs. Let v be a game with m MWCs. Then $v = v_1 \vee \ldots \vee v_m$, where each v_i, is an unanimity game on the ith MWC.

Let A be the set of players that do not belong to every MWC, i.e. $A = \{i \in N \mid \exists\ T$ a MWC, $\ni i \notin T\}$. Let B be the set of players that belong to all MWCs. So B = N - A.

$\forall i \in A$, let T be a MWC $\ni i \notin T$. Then the unanimity game v_T is one of the games v_j. Let $v' = v_1 \vee \ldots \vee v_{j-1} \vee v_{j+1} \vee \ldots \vee v_m$, the game v without MWC T. Obviously, $v = v' \vee v_T$.

By UTr, $\psi_i(v) = \psi_i(v') - \psi_i(v \wedge u_T)$. Both v′ and $v \wedge u_T = u_T$ are proper and have fewer than m MWCs, so $\psi_i(v')$ and $\psi_i(v \wedge u_T)$ are known. Therefore, we can determine $\psi_i(v)$, $\forall i \in A$.

By ET, all $i \in B$ receive the same payoff. Since we know the payoffs for $i \notin B$, by Eff we known the payoffs for $i \in B$. In fact,

$$\psi_i(v) = \frac{1}{|B|}\left[1 - \sum_{j \in A} \psi_j(v)\right]., \quad \forall\, i \in B.$$

The Banzhaf power index can similarly be characterized using Eff′. The formula for $i \in B$ is then

$$\psi_i(v) = \frac{1}{|B|}\left[\eta'(v) - \sum_{j \in A} \psi_j(v)\right]., \quad \forall\, i \in B.$$

Q.E.D.

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