Central Extensions of Partially Defined Games

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Abstract

Mathematicians use the term cooperative game to describe a situation in which members of a group acquire profit as a result of a joint venture. The known methods for dividing the profits between members (players) require knowing the profits that every possible subset of the group would have earned. A player's portion of the profit is calculated based on the performance of each coalition that he would have been part of. In this paper, we consider several ways of applying a particular popular method to cases when full information about the smaller coalitions is unavailable.

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1. Introduction

Meet Alan, Bob and Casper. A, B and C, as we will refer to them from now on, are three thugs who have just completed their jail sentences. All of them were used to working in a team and in their first attempts to operate solo, each found that he could not garner a single cent of profit. So A teamed up with B and at the end of their first week, they had \$3000 to split between the two of them. B decided to take a vacation and A, for a time, formed a coalition with C, resulting in a profit of \$2000. When B got back, A decided to go visit his grandmother, leaving B and C to work together. However, this coalition was less successful than the previous two, garnering no profit whatsoever. When A got back, he decided to form the *Grand Coalition*. Within the first week of working together, the three thugs made \$5000! Should they split it evenly? Had not past experience shown that, without A's great mind, B and C were worth nothing? Not wanting to cause disputes within the Grand Coalition, the thugs kidnapped a mathematician to divide the money for them with utmost fairness.

The first thing the mathematician did was obtain all the information on the *worths* of the smaller coalitions (that is, how much money they made), including those he called single-player coalitions. He called them all *players* now.

coalition	worth
A	0
B	0
C	0
AB	3000
AC	2000
BC	0
ABC	5000

This was a lot like making a value table for a function, except that the domain of this function, which the mathematician called a *cooperative game* and dubbed v for short, was not the real numbers but rather the set of all possible coalitions of players. The mathematician then jotted down the requirements that the players had for his method of allocation. First, it had to be a formula that would work for any amount of money and that would split the entire sum among A,B and C. Second, should it become known that the coalition AB, for example, had actually made more than \$3000, the sum allocated to either of these players should not become smaller. An allocation method for cooperative games known as the *Shapley formula* satisfies these conditions. It is based on each player's marginal contribution to every coalition he is part of. In other words it compares the worth of every coalition S involving player i with the worth of the coalition $S\backslash\{i\}$. The portion of the profits given to the player, his *Shapley value*, is defined as

$$\varphi_i = \sum_{S\subseteq N, i\in S} \frac{(n-s)!(s-1)!}{n!}[v(S) - v(S\backslash\{i\})] \tag{1}$$

where N is the set of all players and n, s are the the number of players in the grand coalition and in coalition S, respectively. In our game[1],

$$\begin{aligned}
\varphi_A &= \frac{1}{3}[v(ABC) - v(BC)] + \frac{1}{6}[v(AB) - v(B)] + \frac{1}{6}[v(AC) - v(C)] = 2500 \\
\varphi_B &= \frac{1}{3}[v(ABC) - v(AC)] + \frac{1}{6}[v(AB) - v(A)] + \frac{1}{6}[v(BC) - v(C)] = 1500 \\
\varphi_C &= \frac{1}{3}[v(ABC) - v(AB)] + \frac{1}{6}[v(AC) - v(A)] + \frac{1}{6}[v(BC) - v(B)] = 1000
\end{aligned}$$

So, mathematicians know of a way to divide any amount between any number of players in a consistent and fair manner. What is the purpose of this research then? The problem is that we do not always know the worth of every possible coalition within a game. We can make estimations of these worths, but in a large game this can become a long and costly process, since the number of possible coalitions grows exponentially with the number of players. And to make the allocation using the Shapley formula we need to know each and every one. This paper is concerned with finding allocations for *partially defined games* (sometimes shortened PDGs), cooperative games defined only over coalitions of certain sizes. Associated with each PDG w is a set J containing the sizes of coalitions whose worths are known. This terminology and notation were first introduced by Letcher in 1990[2]. The types of PDGs you will see in this paper include even-player games in which the worths of all coalitions involving exactly half of the players are unknown ($J = N\backslash\{\frac{n}{2}\}$), and games where all worths are unknown except for the grand coalition, single-player coalitions and the coalitions involving all players but one ($J = \{1, n-1, n\}$). All games we will look at are *zero-normalized*, that is, reduced to a form where all single-player coalitions are worth 0.

[1] There is another term, $v(A) - v(\emptyset)$, but since the worths of all single player coalitions in this game (and in all other games we will consider) are 0, we have eliminated it from these equations.

[2] Letcher D.(1990) The Shapley Value on Partially Defined Games. Research Experiences for Undergraduates Final Report

Our method of dealing with these games will be first to impose a reasonable restriction on the unknown worths. For example, we can assume that our game is *monotonic.* That is, adding another player to any of the coalitions does not bring down the worth of that coalition or, more formally,

$$v(S) \leq v(S + \{i\}), \text{ for all } i \in N \backslash S, S \subset N \tag{2}$$

A *superadditive* game implies an even stronger restriction - cooperation is never harmful. Any two coalitions joining forces are worth at least as much as the sum of their profits before the merger:

$$v(P) + v(Q) \leq v(P \cup Q) \text{ for all } P, Q \subseteq N, P \cap Q = \varnothing \tag{3}$$

An *extension* of a PDG is a fully defined game that takes the known worths from the PDG and assigns some values to the unknown worths. Making an assumption of monotonicity or superadditivity confines all "legitimate" extensions of the PDG within a convex region of the $(2^n - 1)$-dimensional space of coalitions. The approach used throughout the first three sections of this paper is finding the "central" extension in this region and applying the Shapley formula to it. Various centers are considered, among them the center of mass, the coordinate center and the Chebyshev center.

In the first three sections of the paper, we find the centroid and coordinate centers of even-player superadditive games with $J = N \backslash \{\frac{n}{2}\}$. Section 4 develops an algorithm for approximating the centroid of a monotonic game with $J = N \backslash \{k-1, k\}$ where $2 < k < n$. The last two sections deal with monotonic PDGs with $J = \{1, n-1, n\}$. In section 5, we define the Chebyshev center extension and relate it to the coordinate extrema center found by another author. In section 6, we propose a new method for finding a fair allocation on a partially defined game. Instead of finding a central extension, we apply the Shapley formula to the entire region, mapping it from the space of coalitions into a region in the lower-dimensional space of players. We then use our notion of fairness to find a central point within that region representing our final solution.

2. Centroid for J=N\{$\frac{n}{2}$}

Definition 1. *The center of mass or centroid of a convex region in $\mathbb{R}^m$ is a point $\bar{x} \in \mathbb{R}^m$ such that*

$$\bar{x}_i = \frac{\int\int_D ... \int x_i dx_1 dx_2 ... dx_m}{V} \quad (4)$$

where V denotes the volume of the region and the limits of integration are its borders.

Lemma 1. *Suppose $x \in \mathbb{R}^m$ is the centroid of a convex subset A of $\mathbb{R}^m$ and $y \in \mathbb{R}^n$ is the centroid of a convex subset B of $\mathbb{R}^n$. Then (x, y) is the centroid of*

$$A \times B = \{(a, b) : a \in A \text{ and } b \in B\}.$$

Proof. The ith coordinate of the centroid of $A \times B$ for $1 \leq i \leq m$ is

$$\begin{aligned}\frac{\int \cdots \int_{A\times B} a_i \, da_1 \cdots da_m db_1 \cdots db_n}{\int \cdots \int_{A\times B} da_1 \cdots da_m db_1 \cdots db_n} &= \frac{\int \cdots \int_A a_i \, da_1 \cdots da_m \ \int \cdots \int_B db_1 \cdots db_n}{\int \cdots \int_A da_1 \cdots da_m \ \int \cdots \int_B db_1 \cdots db_n} \\ &= \frac{\int \cdots \int_A a_i \, da_1 \cdots da_m}{\int \cdots \int_A da_1 \cdots da_m}\end{aligned}$$

which is the ith coordinate of the centroid of A. A similar calculation shows that the $(m+i)$th coordinate of the centroid of $A \times B$ for $1 \leq i \leq n$ is the ith coordinate of the centroid of B.

Lemma 2. *Suppose $0 \leq x_0 \leq 1$ and $0 \leq y_0 \leq 1$. Let*

$$C = \{(x, y) : 0 \leq x \leq x_0, 0 \leq y \leq y_0, x + y \leq 1\}.$$

Then the centroid of C is the point $[\bar{x}, \bar{y}]$ such that, for $x_0 + y_0 < 1$, $\bar{x} = \frac{x_0}{2}$ and $\bar{y} = \frac{y_0}{2}$. For $x_0 + y_0 \geq 1$,

$$\begin{aligned}\overline{x} &= \frac{1}{3}\frac{3x_0^2 - 2x_0^3 - (1-y_0)^3}{2x_0 - x_0^2 - (1-y_0)^2} \text{ and} \\ \overline{y} &= \frac{1}{3}\frac{3y_0^2 - 2y_0^3 - (1-x_0)^3}{2y_0 - y_0^2 - (1-x_0)^2}\end{aligned}$$

Proof.

Case 1: $x_0 + y_0 \leq 1$. $x + y \leq 1$ holds for any $x, y \in C$ so that C can now be written as $\{(x, y) : 0 \leq x \leq x_0, 0 \leq y \leq y_0\}$, a rectangle with side lengths x_0, y_0. The centroid of a rectangle is $\bar{x} = \frac{x_0}{2}$ and $\bar{y} = \frac{y_0}{2}$.

Case 2: $x_0 + y_0 > 1$. C is a rectangle minus an isosceles right triangle. Adapting the definition of centroid given above (4), the centroid of a two-dimensional region is given by $[\bar{x}, \bar{y}]$ such that

$$\overline{x} = \frac{\int\int_D x\,dy\,dx}{A}$$

$$\overline{y} = \frac{\int\int_D x\,dy\,dx}{A}$$

The area of C for $x_0 + y_0 \geq 1$ is

$$A_C = x_0 y_0 - \frac{(x_0 + y_0 - 1)^2}{2}$$

Substituting A_C and the boundaries of C into the above definition, we obtain

$$\overline{x} = \frac{\int_0^{1-y_0}\int_0^{y_0} x\,dy\,dx + \int_{1-y_0}^{x_0}\int_0^{1-x} x\,dy\,dx}{x_0 y_0 - \frac{(x_0+y_0-1)^2}{2}}$$

$$\overline{y} = \frac{\int_0^{1-y_0}\int_0^{y_0} y\,dy\,dx + \int_{1-y_0}^{x_0}\int_0^{1-x} y\,dy\,dx}{x_0 y_0 - \frac{(x_0+y_0-1)^2}{2}}$$

The numerator of $\overline{x}$ expands to

$$\begin{aligned}
&\int_0^{1-y_0} (x\,y_0)\,dx + \int_{1-y_0}^{x_0} x\,(1-x)\,dx \\
= \;& y_0 \frac{(1-y_0)^2}{2} + \frac{x_0^2}{2} - \frac{x_0^3}{3} - \frac{(1-y_0)^2}{2} + \frac{(1-y_0)^3}{3} \\
= \;& \frac{(1-y_0)^3}{3} - \frac{(1-y_0)^3}{2} + \frac{x_0^2}{2} - \frac{x_0^3}{3} \\
= \;& \frac{3x_0^2 - 2x_0^3 - (1-y_0)^3}{6}
\end{aligned}$$

Similarly, the numerator of $\overline{y}$ becomes

$$\frac{3y_0^2 - 2y_0^3 - (1-x_0)^3}{6}$$

Expanding the denominator and simplifying, we obtain

$$\overline{x} = \frac{1}{3}\frac{3x_0^2 - 2x_0^3 - (1-y_0)^3}{2x_0 - x_0^2 - (1-y_0)^2}$$

$$\overline{y} = \frac{1}{3}\frac{3y_0^2 - 2y_0^3 - (1-x_0)^3}{2y_0 - y_0^2 - (1-x_0)^2}$$

QED

Definition 2. *For a partially defined game w define*

$$w^{\min}(S) = \max\{w(P) + w(Q) : P, Q \subset N, P \cup Q = S, P \cap Q = \varnothing, \{p, q\} \subseteq J\}$$

$$w^{\max}(S) = \min\{w(S \cup Q) - w(Q) : Q \subset N \backslash S, q \in J\}$$

Theorem 3. *Suppose w is a zero-normalized partially defined game such that $J = N\backslash\{n/2\}$ where n is even. If $\overline{w}$ is the superadditive centroid extension of game w then, for all $|S| \notin J$*

$$\overline{w}(S) = \begin{cases} w^{\min}(S) + l(S)/2 \text{ if } l(S) + l(N\backslash S) \leq \varepsilon(S) \\ w^{\min}(S) + \frac{1}{3}\frac{3\varepsilon(S)l(S)^2 - 2l(S)^3 - [\varepsilon(S) - l(N\backslash S)]^3}{2\varepsilon(S)l(S) - l(S)^2 - [\varepsilon(S) - l(N\backslash S)]^2} \\ \qquad \text{if } l(S) + l(N\backslash S) > \varepsilon(S) \end{cases}$$

where

$$l(S) = w^{max}(S) - w^{min}(S)$$

and

$$\varepsilon(S) = \varepsilon(N\backslash S) = w(N) - w^{\min}(S) - w^{\min}(N\backslash S).$$

Proof. We claim that $\breve{w}$ is a superadditive extension of w if and only if, for all $S \subset N$,

$$\breve{w}(S) = w(S) \text{ for all } s \neq \frac{n}{2} \tag{5}$$

and

$$w^{min}(S) \leq \breve{w}(S) \leq w^{max}(S) \tag{6}$$

and

$$\breve{w}(S) + \breve{w}(N\backslash S) \leq w(N) \tag{7}$$

Indeed, suppose (5), (6), and (7) hold for all $S \subset N$. By (5), $\breve{w}$ is an extension of w. Suppose $P, Q \subset N$ satisfies $P \cap Q = \emptyset$. To show that $\breve{w}$ is superadditive, we will show that $\breve{w}(P) + \breve{w}(Q) \leq \breve{w}(P \cup Q)$. By (6), if $q \in J$, then $\breve{w}(P) \leq w^{\max}(P) \leq \breve{w}(P \cup Q) - \breve{w}(Q)$ which implies $\breve{w}(P) + \breve{w}(Q) \leq \breve{w}(P \cup Q)$. Similary, if $p \in J$, then $\breve{w}(Q) \leq w^{\max}(Q) \leq \breve{w}(Q \cup P) - \breve{w}(P)$ which implies $\breve{w}(P) + \breve{w}(Q) \leq \breve{w}(P \cup Q)$. Finally, if $p, q \notin J$, then $p = q = \frac{n}{2}$ and $Q = N \backslash P$ (because $P \cap Q = \emptyset$), and so (7) implies $\breve{w}(P) + \breve{w}(Q) \leq \breve{w}(P \cup Q)$. Thus, $\breve{w}$ is a superadditive extension of w.

Conversely, suppose $\breve{w}$ is a superadditive extension of w. Suppose $S \subset N$. Condition (5) holds because $\breve{w}$ an extension, and (7) follows directly from superadditivity of $\breve{w}$. Since $\breve{w}$ is superadditive, $\breve{w}(S) \leq \breve{w}(S \cup Q) - \breve{w}(Q)$ for all $Q \subset N \backslash S$ which implies $\breve{w}(S) \leq w^{max}(S)$, and $\breve{w}(S) \geq w(P) + w(Q)$ for all $P, Q \subset N$ satisfying $P \cup Q = S$ and $P \cap Q =$ which implies $w^{min}(S) \leq \breve{w}(S)$. So, (6) holds.

As we have just seen, the only coalition of unknown worth affecting the worth of coalition $S, s = \frac{n}{2}$ is the coalition of all other players $N \backslash S$. We can, therefore, separate all coalitions of size $n/2$ into pairs and look at the region Ω containing all superadditive extensions of w as a direct product of $\frac{N!}{(N/2)!^2}$ regions of the form.

$$\begin{aligned}
\vartheta &= \{(x, y) : w^{min}(S) \leq x \leq w^{max}(S), w^{min}(N \backslash S) \leq y \leq w^{max}(N \backslash S), x + y \leq w(N)\} \\
&= \{(w^{\min}(S) + x, w^{\min}(N \backslash S) + y) : 0 \leq x \leq l(S), 0 \leq y \leq l(N \backslash S), x + y \leq \varepsilon(S)\} \\
&= \{(w^{\min}(S) + \varepsilon(S)x, w^{\min}(N \backslash S) + \varepsilon(S)y) : \qquad (8) \\
&\qquad 0 \leq x \leq l(S)/\varepsilon(S), 0 \leq y \leq l(N \backslash S)/\varepsilon(S), x + y \leq 1\} \qquad (9)
\end{aligned}$$

Being a center of mass, the centroid of a region is covariant with respect to translations and scale changes. So, we can find the centroid of ϑ by using the centroid of the region C from Lemma 2, setting $x_0 = l(S)/\varepsilon(S)$ and $y_0 = l(N \backslash S)/\varepsilon(S)$, and then scaling the result by the factor $\varepsilon(S)$ and shift it right by $w^{\min}(S)$ and up by $w^{\min}(N \backslash S)$. Hence, the S coordinate of the centroid of ϑ is

$$\begin{aligned}
&\text{For } l(S) + l(N \backslash S) \leq \varepsilon(S), \quad \overline{w}(S) = w^{\min}(S) + l(S)/2 \\
&\text{For } l(S) + l(N \backslash S) > \varepsilon(S), \quad \overline{w}(S) = w^{\min}(S) + \\
&\qquad + \frac{1}{3} \frac{3\varepsilon(S)l(S)^2 - 2l(S)^3 - [\varepsilon(S) - l(N \backslash S)]^3}{2\varepsilon(S)l(S) - l(S)^2 - [\varepsilon(S) - l(N \backslash S)]^2}
\end{aligned}$$

QED

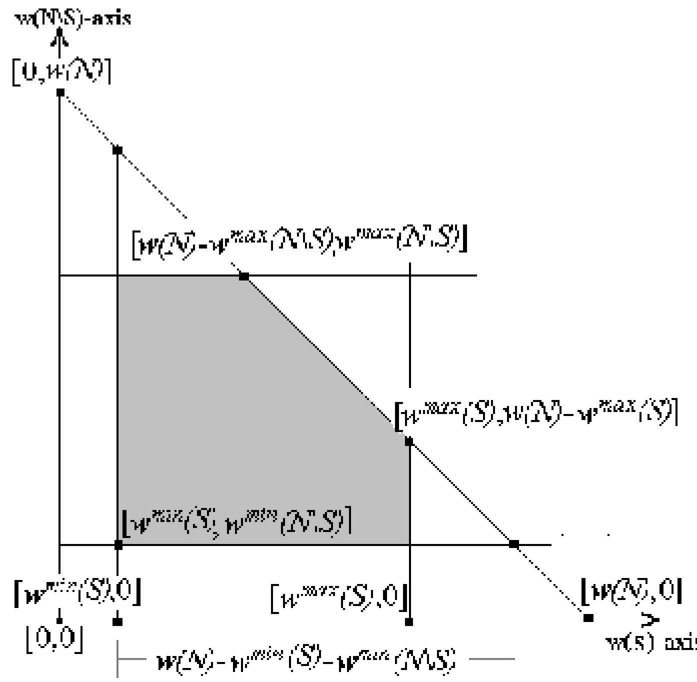

Figure 1. Visual representation of region ϑ for $l(S) + l(N\backslash S) > \varepsilon(S)$

3. Coordinate Center for J=N\{N/2}

Definition 3. *A coordinate center of a convex set $C \subset \mathbb{R}^m$ is a vector x which is the midpoint of $\{x+\lambda e_i : \lambda \in \mathbb{R}\} \cap C$ for all $i = 1, 2, \ldots, m$ where e_i denotes a vector in $\mathbb{R}^m$ such that it's ith element is 1 and all other elements are 0.*

Definition 4. *For $u, v \in \mathbb{R}^m$ and $C \subseteq \mathbb{R}^m$, let the element-wise product of u and v, $(u * v)$, be a vector w such that $w_i = (u * v)_i = u_i v_i$. Define $u * C$ as $C' = \{u * x : x \in C\}$.*

In the following lemma, we show that coordinate centers are covariant with respect to scale changes and translations.

Lemma 4. *Let C be a convex set in $\mathbb{R}^m$ and let $a, b \in R^m$, where all elements of a are non-zero. If $x \in \mathbb{R}^m$ is a coordinate center of C, then $a * x + b$ is a coordinate center of $a * C + b$*

Proof. By definition of a coordinate center (see above), x is the midpoint of the line segment $Y_i = \{x + \lambda e_i : \lambda \in \mathbb{R}\} \cap C$ for all $i = 1, ..., m$. We need to show that $z = a * x + b$ is the midpoint of all line segments of the form $\{z + \eta e_i : \eta \in \mathbb{R}\} \cap a * C + b$. Let $\eta = \lambda a_i$. Since a_i is non-zero and $\mathbb{R}$ is closed under multiplication, $\lambda \in \mathbb{R}$ implies $\lambda a_i \in \mathbb{R}$ and

$$\begin{aligned}
\{z + \eta e_i \quad : \quad \eta \in \mathbb{R}\} &= \{a * x + b + \lambda a_i e_i : \lambda a_i \in \mathbb{R}\} \\
&= \{a * x + b + a_i(\lambda e_i) : \lambda \in \mathbb{R}\} \\
&= \{a * x + a * \lambda e_i + b : \lambda \in \mathbb{R}\} \\
&= \{a * (x + \lambda e_i) : \lambda \in \mathbb{R}\} + b \\
&= a * \{x + \lambda e_i : \lambda \in \mathbb{R}\} + b
\end{aligned}$$

for all i. So $\{z + \eta e_i : \eta \in \mathbb{R}\} \cap a * C + b = a * Y_i + b$. The midpoint of the scaled and translated segment will be the scaled and translated midpoint of the original segment, $z = a * x + b$. QED

Lemma 5. *Suppose $x \in \mathbb{R}^m$ is the coordinate center of a convex subset A of $\mathbb{R}^m$ and $y \in \mathbb{R}^n$ is the coordinate center of a convex subset B of $\mathbb{R}^n$. Then (x, y) is the coordinate center of*

$$A \times B = \{(a, b) : a \in A \text{ and } b \in B\}.$$

Proof. Follows directly from the definition of the coordinate center.

Lemma 6. *Suppose $0 \leq x_0 \leq 1$ and $0 \leq y_0 \leq 1$. Define C as in the previous chapter:*

$$C = \{(x, y) : 0 \leq x \leq x_0, 0 \leq y \leq y_0, x + y \leq 1\}.$$

Then

$$[x', y'] = \begin{cases} [\frac{x_0}{2}, \frac{y_0}{2}] & \text{if } y_0 < 1 - \frac{x_0}{2} \text{ and } x_0 < 1 - \frac{y_0}{2} \\ [\frac{1-\frac{y_0}{2}}{2}, \frac{y_0}{2}] & \text{if } y_0 < \frac{2}{3} \text{ and } x_0 \geq 1 - \frac{y_0}{2} \\ [\frac{x_0}{2}, \frac{1-\frac{x_0}{2}}{2}] & \text{if } x_0 < \frac{2}{3} \text{ and } y_0 \geq 1 - \frac{x_0}{2} \\ [\frac{1}{3}, \frac{1}{3}] & \text{if } y_0 \geq \frac{2}{3} \text{ and } x_0 \geq \frac{2}{3} \end{cases}$$

is the unique coordinate center of C.

Proof. We first show that the point defined in the lemma is a coordinate center of C.

1. If $y_0 < 1 - \frac{x_0}{2}$ and $x_0 < 1 - \frac{y_0}{2}$ then $[x', y'] = [\frac{x_0}{2}, \frac{y_0}{2}]$.

Since $y_0 < 1 - \frac{x_0}{2}$ a line through $[x', y']$ parallel to the y-axis will intersect the boundary at point $p_1 = [\frac{x_0}{2}, y_0]$ and not $[\frac{x_0}{2}, 1 - \frac{x_0}{2}]$. The same line will intersect the boundary at point $p_2 = [\frac{x_0}{2}, 0]$. Similarly, since $x_0 < 1 - \frac{y_0}{2}$ a line parallel to the x-axis will intersect the boundary at $p_3 = [x_0, \frac{y_0}{2}]$ and $p_4 = [0, \frac{y_0}{2}]$. The length of the line segment connecting $[x', y']$ and p_1 is equal to length of the line segment connecting $[x', y']$ and p_2 and the segments connecting $[x', y']$ to p_3 and p_4 are also equal in length. By definition of a coordinate center $[x', y']$ is a coordinate center of C.

2. If $y_0 < \frac{2}{3}$ and $x_0 \geq 1 - \frac{y_0}{2}$ then $[x', y'] = [\frac{1-\frac{y_0}{2}}{2}, \frac{y_0}{2}]$ is a coordinate center of C.

Since $x_0 \geq 1 - \frac{y_0}{2}$, a line through $[\frac{1-\frac{y_0}{2}}{2}, \frac{y_0}{2}]$ parallel to the x-axis will intersect the boundary at $[1 - \frac{y_0}{2}, \frac{y_0}{2}]$ as well as $[0, \frac{y_0}{2}]$. The distance from $[x', y']$ to both points is $\frac{1-\frac{y_0}{2}}{2}$. $y_0 < \frac{2}{3}$ implies $\frac{3}{2}y_0 < 1$ which implies $2y_0 < 1 + \frac{y_0}{2}$ implying $2y_0 < 2 - (1 - \frac{y_0}{2})$ which in turn implies $y_0 < 1 - \frac{1-\frac{y_0}{2}}{2}$. Therefore, a line through $[x', y']$ parallel to the y-axis will intersect the boundary at $[x', y_0]$. The distances between $[x', y']$ and $[x', 0]$ and between $[x', y']$ and $[x', y_0]$ are equal. By definition, $[x', y']$ is a coordinate center of C.

3. If $x_0 < \frac{2}{3}$ and $y_0 \geq 1 - \frac{x_0}{2}$ then $[x', y'] = [\frac{x_0}{2}, \frac{1-\frac{x_0}{2}}{2}]$ is a coordinate center of C.

Interchanging the x' with y' and x_0 with y_0 in the proof for case 2, we obtain the proof for case 3.

4. If $y_0 \geq \frac{2}{3}$ and $x_0 \geq \frac{2}{3}$ then $[x', y'] = [\frac{1}{3}, \frac{1}{3}]$ is a coordinate center of C.

Since $y_0 \geq 1 - x' = \frac{2}{3}$ a line through $[x', y']$ parallel to the y-axis will intersect the boundary at $[\frac{1}{3}, 0]$ and $[\frac{1}{3}, \frac{2}{3}]$. Similarly, a line through $[x', y']$ parallel to the x-axis will intersect the boundary at $[0, \frac{1}{3}]$ and $[\frac{2}{3}, \frac{1}{3}]$.The distance from

$[x', y']$ to each of these points is $\frac{1}{3}$. According to the definition, $[x', y']$ is a coordinate center of C.

To prove uniqueness let $[x'', y'']$ be a coordinate center of C. We will show that $[x'', y''] = [x', y']$.

If $1-y'' > x_0$ then a line through $[x'', y'']$ parallel to the x-axis will intersect the boundary of C at $[0, y'']$ and at $[x_0, y'']$ before $[1-y'', y'']$ which implies $x'' = \frac{x_0}{2}$, the midpoint. If, on the other hand, $1-y'' \leq x_0$ then the line will intersect the boundary of C at $[0, y'']$ and $[1-y'', y'']$ which implies $x'' = \frac{1-y''}{2}$. Similarly, if $1-x'' > y_0$,then a $y'' = \frac{y_0}{2}$ and otherwise $y'' = \frac{1-x''}{2}$.

Let us look at four cases:

If $1-y'' > x_0$ and $1-x'' > y_0$ then $x'' = \frac{x_0}{2}$ and $y'' = \frac{y_0}{2}$. Substituting these values back into the inequalities,we get $y_0 < 1 - \frac{x_0}{2}$ and $x_0 < 1 - \frac{y_0}{2}$.

If $1-y'' \leq x_0$ and $1-x'' > y_0$ then $y'' = \frac{y_0}{2}$ and $x'' = \frac{1-y''}{2} = \frac{1-\frac{y_0}{2}}{2}$. Substituting $\frac{y_0}{2}$ into the first inequality get $x_0 \geq 1 - \frac{y_0}{2}$. Substituting $\frac{1-\frac{y_o}{2}}{2}$ into the second inequality get $y_0 < 1 - \frac{1-\frac{y_0}{2}}{2} \implies 2y_0 < 1 + \frac{y_0}{2} \implies \frac{3}{2}y_0 < 1 \implies y_0 < \frac{2}{3}$.

Similarly if $1-y'' > x_0$ and $1-x'' \leq y_0$, $x'' = \frac{x_0}{2}$ and $y'' = \frac{1-\frac{x_o}{2}}{2}$. Substituting back, get $x_0 < \frac{2}{3}$ and $y_0 \geq 1 - \frac{x_0}{2}$.

If $1-y'' \leq x_0$ and $1-x'' \leq y_0$ then $x'' = \frac{1-y''}{2}$ and $y'' = \frac{1-x''}{2}$. Substituting $\frac{1-x''}{2}$ for y'' into the first equation, get $x'' = \frac{1}{3}$. Substituting $\frac{1}{3}$ for x'' into the second equation, get $y'' = \frac{1}{3}$. Substituting $x'' = y'' = \frac{1}{3}$ get $y_0 \geq \frac{2}{3}$ and $x_0 \geq \frac{2}{3}$
.

$[x', y'] = [x'', y'']$ for all values of y_0 and x_0. Therefore $[x', y']$ is a unique coordinate center of C.

QED

Theorem 7. *Suppose w is a zero-normalized, superadditive partially defined game such that $J = N\backslash\{n/2\}$ where $n = |N|$ is even. If $\breve{w}$ is the coordinate center extension of w then, for all $S \subseteq N, |S| = \frac{n}{2}$*

$$\breve{w}(S) = \begin{cases} w^{min}(S) + \frac{1}{2}\varepsilon(S) - \frac{1}{4}l(N\backslash S) & \text{if } l(S) < \frac{2}{3}\varepsilon(S) \text{ and } l(N\backslash S) \geq \varepsilon(S) - \frac{l(S)}{2} \\ w^{min}(S) + \frac{1}{3}\varepsilon(S) & \text{if } l(S) \geq \frac{2}{3}\varepsilon(S) \text{ and } l(N\backslash S) \geq \frac{2}{3}\varepsilon(S) \\ w^{min}(S) + \frac{l(S)}{2} & \text{elsewhere} \end{cases}$$

where

$$\varepsilon(S) = \varepsilon(N\backslash S) = w(N) - w^{\min}(S) - w^{\min}(N\backslash S)$$

and

$$l(S) = w^{max}(S) - w^{min}(S).$$

Proof.

As we have shown in the proof to Theorem 1, the region Ω containing all superadditive extensions of w can be viewed as a direct product of the regions

$$\vartheta_k = \{(w^{\min}(T_k) + \varepsilon(T_k)x, w^{\min}(N\backslash T_k) + \varepsilon(T_k)y) : \\ 0 \le x \le l(T_k)/\varepsilon(T_k), 0 \le y \le l(N\backslash T_k)/\varepsilon(T_k), x + y \le 1\}, k = 1...d$$

where $d = \frac{N!}{2(N/2)!^2}$ is the number of distinct $T, N/T$ pairs. Each region defines two dimensions of Ω corresponding to coalitions $T, N/T$.

Let

$$\begin{aligned} \alpha &= [1/\varepsilon(T_1), 1/\varepsilon(T_1), 1/\varepsilon(T_2), 1/\varepsilon(T_2), ..., 1/\varepsilon(T_d), 1/\varepsilon(T_d)] \\ \beta &= [w^{\min}(T_1), w^{\min}(N\backslash T_1), ..., w^{\min}(T_d), w^{\min}(N\backslash T_d)] \\ F(S) &= l(S)/\varepsilon(S) \end{aligned}$$

Let

$$\Omega' = \alpha * (\Omega - \beta)$$

Ω' is a direct product of

$$\vartheta'_k = \{(x, y) : 0 \le x \le F(S), 0 \le y \le F(N\backslash S), x + y \le 1\}, k = 1...d$$

According to Lemmas 4 and 5, the coordinate center of Ω' is a vector $x \in \mathbb{R}^{2*d}$ such that, for every $k = 1, ..., d$, $(x_{2k-1}, x_{2k}) = (x_{T_k}, x_{N\backslash T_k}) =$

$$\begin{cases} [\frac{F(T_k)}{2}, \frac{F(N\backslash T_k)}{2}] & \text{if } F(N\backslash T_k) < 1 - \frac{F(T_k)}{2} \text{ and } F(T_k) < 1 - \frac{F(N\backslash T_k)}{2} \\ [\frac{1-\frac{F(N\backslash T_k)}{2}}{2}, \frac{F(N\backslash T_k)}{2}] & \text{if } F(N\backslash T_k) < \frac{2}{3} \text{ and } F(T_k) \ge 1 - \frac{F(N\backslash T_k)}{2} \\ [\frac{F(T_k)}{2}, \frac{1-\frac{F(T_k)}{2}}{2}] & \text{if } F(T_k) < \frac{2}{3} \text{ and } F(N\backslash T_k) \ge 1 - \frac{F(T_k)}{2} \\ [\frac{1}{3}, \frac{1}{3}] & \text{if } F(N\backslash T_k) \ge \frac{2}{3} \text{ and } F(T_k) \ge \frac{2}{3} \end{cases}$$

Since $N\backslash(N\backslash T) = T$, we can rewrite this as

$$x_S = \begin{cases} \frac{1-\frac{F(N\backslash S)}{2}}{2} & \text{if } F(N\backslash S) < \frac{2}{3} \text{ and } F(S) \ge 1 - \frac{F(N\backslash S)}{2} \\ \frac{1}{3} & \text{if } F(N\backslash S) \ge \frac{2}{3} \text{ and } F(S) \ge \frac{2}{3} \\ \frac{F(S)}{2} & \text{elsewhere} \end{cases}$$

Let α' be a vector such that $\alpha'_i = 1/\alpha_i$ for all $i = 1...2d$. According to Lemma 3, the coordinate center of $\Omega = \alpha' * \Omega' + b$ is defined by $\breve{w} = \alpha' * x + b$. Performing the scale change and translation and simplifying, we obtain

$$\breve{w}(S) = \begin{cases} w^{min}(S) + \frac{1}{2}\varepsilon(S) - \frac{1}{4}l(N\backslash S) & \text{if } l(S) < \frac{2}{3}\varepsilon(S) \text{ and } l(N\backslash S) \ge \varepsilon(S) - \frac{l(S)}{2} \\ w^{min}(S) + \frac{1}{3}\varepsilon(S) & \text{if } l(S) \ge \frac{2}{3}\varepsilon(S) \text{ and } l(N\backslash S) \ge \frac{2}{3}\varepsilon(S) \\ w^{min}(S) + \frac{l(S)}{2} & \text{elsewhere} \end{cases}$$

QED.

4. Distance Between Centers of J=N\{n/2}

The following theorem is useful when the coordinate center is used as an approximation of the centroid of the game.

Theorem 8. *If $\overline{w}$ is the centroid extension of game w where $J = N\backslash\{n/2\}$ and $\breve{w}$ is its coordinate center extension then the Euclidian distance between $\overline{w}$ and $\breve{w}$ is less than or equal to*

$$E^{\max} = \frac{1}{72}\sqrt{10 \sum_{S, s=n/2} \varepsilon(S)^2}$$

where $\varepsilon(S) = \varepsilon(N\backslash S) = w(N) - w^{\min}(S) - w^{\min}(N\backslash S)$.

Proof. As we have already shown, the set of superadditive extensions of w can be represented as the direct product of sets of the form

$$\begin{aligned} \vartheta &= \{(w^{\min}(T) + \varepsilon(T)x, w^{\min}(N\backslash T) + \varepsilon(T)y) : \\ 0 &\leq x \leq l(T)/\varepsilon(T), 0 \leq y \leq l(N\backslash T)/\varepsilon(T), x + y \leq 1\} \end{aligned}$$

The centroid and coordinate center extensions of w were found by considering the centroid and coordinate center of

$$C = \{(x, y) : 0 \leq x \leq x_0, 0 \leq y \leq y_0, x + y \leq 1\}.$$

So it seems logical that we should now revert back to this set and investigate the distance between its centroid and coordinate center. A detailed investigation of extrema points of the function

$$D = \sqrt{(x' - \bar{x})^2 + (y' - \bar{y})^2}$$

(see Appendix) reveals a maximum of $\frac{\sqrt{5}}{36}$ at $\left[1, \frac{2}{3}\right]$ and $\left[\frac{2}{3}, 1\right]$ where

$$D = \sqrt{\frac{1}{36^2} + \frac{4}{36^2}}$$

Rescaling by $\varepsilon(S)$ we obtain $\frac{\sqrt{5}}{36}\varepsilon(S) = \sqrt{\left(\frac{1}{36}\varepsilon(S)\right)^2 + \left(\frac{2}{36}\varepsilon(N\backslash S)\right)^2}$ as the maximum distance between the centroid and coordinate center of region ϑ. It follows that for the set of all superadditive extensions, the distance will be

$$\sqrt{\sum \left(\left(\frac{1}{36}\varepsilon(S)\right)^2 + \left(\frac{2}{36}\varepsilon(N\backslash S)\right)^2\right)}$$

where the summation is over all pairs $\{S, N\backslash S\}$. We rewrite this as a summation over all S using the equality of $\varepsilon(S)$ and $\varepsilon(N\backslash S)$.

$$\begin{aligned} & \sqrt{\sum \left(\frac{1}{36}\varepsilon(S)\right)^2 + \sum \left(\frac{2}{36}\varepsilon(N\backslash S)\right)^2} \\ = & \sqrt{\frac{1}{36^2}\sum \varepsilon(S)^2 + \frac{4}{36^2}\sum \varepsilon(S)^2} \\ = & \frac{1}{36}\sqrt{5\sum \varepsilon(S)^2} = \frac{1}{36}\sqrt{\frac{1}{2} * 5 \sum_{S, s=n/2} \varepsilon(S)^2} \end{aligned}$$

QED

5. Approximation of centroid for J=N\{k-1,k}

The algorithm provides a computational approach to finding the centroid of a game where the worths of coalitions of two consecutive sizes are unknown. If we could generate a large number of points randomly distributed within a region, the average of these points can be used as an approximation of the center of mass.

How do you generate the points? Let us take as an example the two-dimensional right triangle in Figure 2. The two shorter sides are 5 units each. In trying to fill it with random points, if we first generated an x-coordinate in the range of 0 to 5, then a y-coordinate in the range of 0 to $5 - x$, we would end up with a far greater concentration of points in the vicinity of vertex B. In order to achieve a good distribution, we need to generate more points with a smaller x-coordinate. Or, we can make our final calculation a weighted average, where a point $[x, y]$ will be weighted based on the length of $5 - x$.

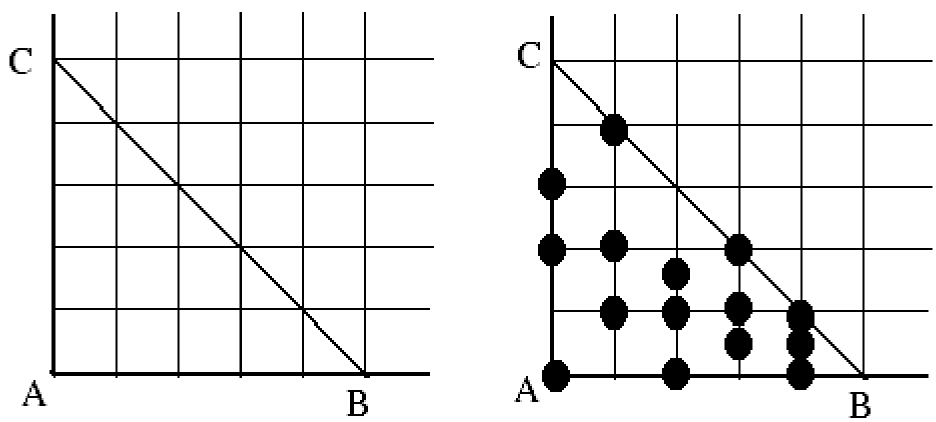

Figure 2. One must be careful when trying to uniformly distribute random points within a region.

The algorithm proceeds like this: given a monotonic partially defined game w where $J_w = N \backslash \{k-1, k\}$, we define $V^{\min}$ and $V^{\max}$ as the top and bottom limits on the worth of S (by monotonicity). Then we randomly generate the worths for all coalitions T of size k in the range from the maximum lower limit among the coalitions of size $k-2$ that are contained in T to the upper limit on T itself, determined by the worths of coalitions of size $k+1$ that contain T. Thus we obtain a partially-defined game $\dot{w}$ where $J_{\dot{w}} = N \backslash \{k-1\}$. We then find the volume and the centroid of the set of monotonic extensions of $\dot{w}$. This is easy because the set is a hyper-cuboid whose side corresponding to a coalition S of size $k-1$ will range from the maximum worth of all coalitions of size$k-2$ contained in S to the minimum worth among all coalitions of size k that S is a subset of. By repeating this procedure a large number of times, we obtain the set of fully defined games $\ddot{w}$ with their associated weights. Our result is obtained by averaging these games.

The Algorithm.

Let w be a zero-normalized, monotonic partially defined game such that $J = N \backslash \{k-1, k\}$.

Let

$$V^{min}(S) = \max_{i \in S} w(S \backslash \{i\})$$

$$V^{max}(S) = \min_{i \in N \backslash S} w(S + \{i\}).$$

Let Z be a large integer. For $t = 1$ to Z do steps 1 - 3.

1. For all $S \subset N$ such that $s = k$,let $V_t(S)$be a uniformly distributed random number between $\max_{i \in S} V^{min}(S \backslash \{i\})$and $V^{max}(S)$.

2. Let $A_t = \prod\limits_{T \subset N, t=k-1} (\min_{i \in N \backslash S} V_t(T \cup \{i\}) - V^{\min}(T))$

3. For all $T \subset N$ such that $t = k-1$,let $V_t(T) = \frac{1}{2}(V^{min}(T) + \min_{i \in N \backslash S} V_t(T \cup \{i\}))$

The centroid extension of w can be approximated by $\bar{w}$ where, for all P of size k or $k-1$,

$$\bar{w}(P) = \frac{\sum\limits_{t=1}^{Z} [A_t \cdot V_t(P)]}{\sum\limits_{t=1}^{Z} A_t}$$

6. Chebyshev center for J={1,n-1,n}

The *Chebyshev center* is defined as the center of the smallest sphere containing all points in the set or a point at which the maximum distance to a point in the set is minimized.

Definition 5. *Let $E(x)$ denote the subset of points in a set Z which have the largest distance from a point x. For an arbitrary set $X \subset \mathbb{R}^m$ the convex hull of X is defined as a set H of all points $h \in \mathbb{R}^m$ such that, for some positive integer k, some points $x_1, x_2, ...x_k$ in X and some positive numbers $t_1, t_2, ...t_k$ such that $\sum_{i=1}^{k} t_i = 1$, $h = \sum_{i=1}^{k} t_i x_i$.*

Lemma 9. *$\tilde{x}$ is the Chebyshev center of a set Z iff $\tilde{x}$ is in the convex hull of $E(\tilde{x})$.*

Proof.
Given by [3]

Definition 6. *Let w denote a partially defined game such that $J = \{1, n-1, n\}$ and $w(\{i\}) = 0$ and $w(N) \geq w(N - \{i\})$ for all $i \in N$. Renumber the players so that $w(N - \{1\}) \leq w(N - \{2\}) \leq ... \leq w(N - \{n\})$. Let a_i denote $w(N - \{i\})$. Let $\eta(S) = min\{i : i \notin S\}$ for all $S \subset N$ such that $|S| \notin J$.*

It is easy to see that

$$0 \leq \ddot{w}(S) \leq a_{\eta(S)} \text{ for all } S \subset N \text{ such that } |S| \notin J \tag{10}$$

is a necessary condition for $\ddot{w}$ to be a monotonic extension of w.

Theorem 10. *If $\tilde{w}$ is the Chebyshev center of the set of monotonic extensions of w then, for all $S \notin J$,*
$\tilde{w}(S) = \frac{1}{2} a_{\eta(S)}$

Proof.
Let $\ddot{w}_1(S) = w(S)$ if $|S| \in J$, $\ddot{w}_1(S) = 0$ if $|S| \notin J$. Let $\ddot{w}_2(S) = w(S)$ if $|S| \in J$, $\ddot{w}_2(S) = a_{i(S)}$ if $|S| \notin J$.

To prove that $\tilde{w}(S)$ is the Chebyshev center of the set of zero-monotonic extensions of w we will show that

1. $\ddot{w}_1(S), \ddot{w}_2(S) \in E(\tilde{w}(S))$ with respect to the set of extensions.

[3] Botkin, N.D, Turova-Botkina, V.L. An algorithm for finding the Chebyshev center of a convex polyhedron. Journal Appl. Math. Optimization, Vol 29, No.2, pp.211-222

2. $\tilde{w}(S)$ can be represented as the convex combination of $\dddot{w}_1(S), \dddot{w}_2(S)$ i.e. for all $S \subset N$, $\tilde{w}(S) = t\dddot{w}_1(S) + (1-t)\dddot{w}_2(S)$ for some positive number t.

Both can be easily demonstrated. Indeed, the distance between $\tilde{w}$ and either $\dddot{w}_1$ or $\dddot{w}_2$ is $\sum\limits_{S\subset N, |S| \notin J} \frac{1}{4}a^2_{i(S)}$. Since, for a given $S, 0 \leq w(S) \leq a_{i(S)}$, the difference between $w(S)$ and $\tilde{w}(S)$ is always less than or equal to $\frac{1}{2}a_{i(S)}$. So $||\tilde{w} - w||^2 \leq \sum\limits_{S\subset N, |S| \notin J} \frac{1}{4}a^2_{i(S)}$. Setting t to $\frac{1}{2}$, obtain, for $|S| \in \{1, n-1, n\}$, $\frac{1}{2}\dddot{w}_1(S) + \frac{1}{2}\dddot{w}_2(S) = \frac{1}{2}w(S) + \frac{1}{2}w(S) = w(S) = \tilde{w}(S)$ by definition of extension. For $S \notin J$, $\frac{1}{2}\dddot{w}_1(S) + \frac{1}{2}\dddot{w}_2(S) = 0 + \frac{1}{2}a_{i(S)} = \tilde{w}(S)$.

Brutt (1994) [4] obtained the same result for the coordinate extrema center of this class of games and found its Shapley value. It is defined by

$$\psi_i = \frac{w(N) - a_n + \frac{1}{2}(a_2 + a_1)}{n} + \sum_{k=2}^{n-1} \frac{2n(n-1) - (n-k-1)(n-k)}{2nk(n-1)}(a_{k+1} - a_k) + \begin{cases} \frac{1}{2}(a_2 - a_1) & \text{if } i = 1 \\ \sum_{m=2}^{n-1} \frac{n-k-1}{2n(n-1)}(a_{k+1} - a_k) & \text{if } i \neq 1 \end{cases}$$

An interesting side note is that, since this result is the same as obtained by Brutt for the coordinate extrema center, the coordinate extrema center of this type of game is equal to the Chebyshev center and therefore is also in the convex hull of the set of extensions. Another observation is that this center is not a center the way we normally see centers – instead of being "in the middle" it situates itself on one of the edges of the polyhedron defined by the monotonic extensions. The monotonicity property is given by the inequalities $w(P) \leq w(Q)$ for any P, Q such that $P \subseteq Q$, while for the Chebyshev center extension, the strict equality $\tilde{w}(P) = \tilde{w}(Q)$ holds for all such P, Q of size between 2 and $n-2$. See Figure 3.

[4] LeeAnne Brutt, A Value for Zero-monotonic Partially Defined Games, manuscript

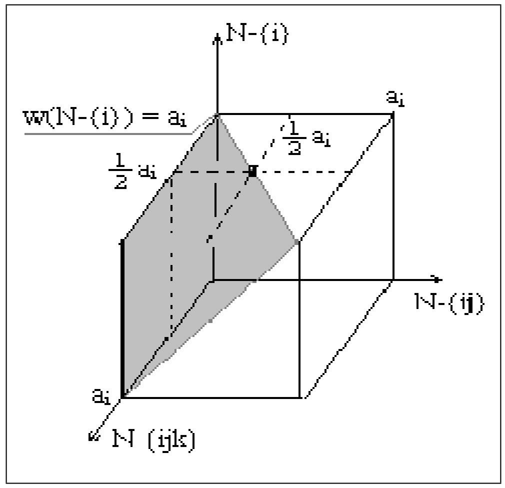

Figure 3. The projection of the set of monotonic extensions into three dimensions $(N - i, N - ij, N - ijk)$

7. A new method for J={1,n-1,n}

In this section, we will look at the highest and the lowest payoff each player could obtain according to the Shapley formula given the known coalition worths. Our notion of fairness says that if a given player obtains a payoff at a certain point between his maximum and minimum payoffs, then no other player should be "closer" or "further", in relative terms, from his maximum payoff. Therefore, we define a fair allocation ξ as

$$\xi_i = \lambda\varphi_i^{\max} + (1-\lambda)\varphi_i^{\min} \tag{11}$$

where $\varphi_i^{\max}, \varphi_i^{\min}$ are the maximum and the minimum possible payoffs to player i and λ is chosen so that

$$\sum_{i=1}^{n} \xi_i = w(N). \tag{12}$$

We will now show how this method can be applied to a game w where $J = \{1, n-1, n\}$ and $w(\{i\}) = 0$ and $w(N) \geq w(N - \{i\}$ for all $i \in N$ and the players are renumbered so that $w(N-\{1\}) \leq w(N-\{2\}) \leq ... \leq w(N-\{n\})$. As in the definition in the previous section, we let a_i denote $w(N - \{i\})$ and let $\eta(S) = min\{i : i \notin S\}$ for all $S \subset N$ such that $|S| \notin J$.

We can rewrite the Shapley formula (equation 1) as

$$\begin{aligned}
\varphi_i(v) &= \sum_{s=1}^{n} \frac{(n-s)!(s-1)!}{n!} \sum_{S\subset N, |S|=s, i\in S} [v(S) - v(S-\{i\})] \\
&= \frac{v(N) - a_i}{n} + \frac{\sum_{k\neq i} a_k - v(N - \{i,k\})}{n(n-1)} \\
&\quad + \frac{\sum_{s=3}^{n-2}(n-s)!(s-1)! \sum_{S,|S|=s,i\in S}[v(S) - v(S-\{i\})]}{n!} \\
&\quad + \frac{\sum_{k\neq i} v(\{i,k\})}{n(n-1)} \\
&= \frac{v(N)}{n} + \frac{\sum_{k=1}^{n}(a_k - a_i)}{n(n-1)} + \frac{\sum_{k\neq i}[v(\{i,k\}) - v(N-\{i,k\})]}{n(n-1)} \\
&\quad + \frac{\sum_{s=3}^{n-2}(n-s)!(s-1)! \sum_{S,|S|=s,i\in S}[v(S) - v(S-\{i\})]}{n!}
\end{aligned}$$

This separates the part of the payoff dependent on the known values and that dependent on the unknown values.

Since $v(S)$ has a positive or negative coefficient in the Shapley formula whenever $i \in S$ or $i \notin S$, respectively, it follows from (10) that the monotonic extension that maximizes $\varphi_i(\hat{w})$ is

$$\hat{w}_i^{\max}(S) = \begin{cases} a_{\eta(S)}, & i \in S \\ 0, & i \notin S \end{cases}$$

for all $S \subset N$ satisfying $|S| \notin J$. The extension giving player i the lowest payoff will minimize the worths of the coalitions that involve i and maximize other worths. Indeed, if we raised the worths of all coalitions not containing i, then we would have the situation where $\hat{w}_i^{\min}(S, i \notin S) = a_{\hat{\imath}(S)} > \hat{w}_i^{\min}(S+\{i\})$. In order not to violate the monotonicity constraint, this extension is defined as

$$\hat{w}_i^{\min}(S) = \begin{cases} a_{\eta(S)}, & i \notin S, |S| = n-2 \\ 0, & \text{otherwise} \end{cases}$$

Substituting $\hat{w}_i^{\max}(S)$ for $v(S)$ into the Shapley formula obtain the maximum payoffs for each player.

$$\varphi_i(\hat{w}_i^{\max}) = \frac{w(N)}{n} + \frac{\sum_{k=1}^{n}(a_k - a_i)}{n(n-1)} + \frac{1}{n(n-1)} * \begin{cases} (n-2)a_2 + a_3, & i = 1 \\ (n-2)a_1 + a_3, & i = 2 \\ (n-2)a_1 + a_2, & i \geq 3 \end{cases} +$$

$$+\frac{1}{n!}\sum_{s=3}^{n-2}\left[(n-s)!(s-1)!\left(\sum_{p=1}^{i-1}\binom{n-p-1}{s-p}a_p + \sum_{q=i+1}^{s+1}\binom{n-q}{s-q+1}a_q\right)\right]$$

The minimum payoffs are

$$\varphi_i(\hat{w}_i^{\min}) = \frac{w(N)}{n} + \frac{\sum(a_k - a_i)}{n(n-1)} - \frac{\sum_{p=1}^{i-1} a_p + (n-i)a_i}{n(n-1)}$$

Solving equations (11) and (12) for λ, we obtain

$$\lambda = \frac{\sum \varphi_i^{\max} - w(N)}{\sum \varphi_i^{\max} - \sum \varphi_i^{\min}}$$

where

$$\sum \varphi_i^{\max} = w(N) + \frac{2(n-2)a_2 + 2a_3 + (n-2)(n-1)a_1}{n(n-1)} + \frac{Sum(w)}{n!}$$

$$\sum \varphi_i^{\min} = w(N) - \frac{1}{n(n-1)}\sum_{i=1}^{n-1} 2(n-i)a_i$$

where $Sum(w)$ can be written as

$$\sum_{i=1}^{n}\sum_{s=3}^{n-2}\left[(n-s)!(s-1)!\left(\sum_{p=1}^{i-1}\binom{n-p-1}{s-p}a_p+\sum_{q=i+1}^{s+1}\binom{n-q}{s-q+1}a_q\right)\right]$$

which, after some manipulation, can be reduced to

$$\sum_{s=3}^{n-2}(n-s)!(s-1)!\sum_{p=1}^{n-1}(n-p+1)\binom{n-p-1}{s-p}a_p$$
$$+\sum_{s=3}^{n-2}(n-s)!(s-1)!\sum_{p=1}^{s}p\binom{n-p-1}{s-p}a_{p+1}$$

Then

$$\lambda=\frac{(n-2)(n-1)a_1+2(n-2)a_2+2a_3+\frac{Sum(w)}{(n-2)!}}{n(n-1)a_1+4(n-2)a_2+(n-1)a_3+\sum_{i=4}^{n-1}2(n-i)a_i+\frac{Sum(w)}{(n-2)!}}$$

Since $\dddot{w}_i^{\max}$ and $\dddot{w}_i^{\min}$ are both superadditive, we have just found an allocation for superadditive as well as monotonic games.