Adding Death to the Fibonacci Sequence

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Introduction

The Fibonacci sequence deals with the reproduction of pairs of rabbits. In the first month, there is one pair of rabbits. They take until the next month to mature to be able to mate. So in the second month, there is still only one pair of rabbits. But in the third month, there are now two pairs of rabbits because they were mature enough to mate with each other. In the fourth month, there will be three pairs of rabbits, because the original pair is mature enough to mate and reproduce while the newer second pair needs this month to mature. And from this sequence of numbers of rabbit pairs, we get the Fibonacci sequence: 1, 1, 2, 3, 5, 8, 13, 21… The Fibonacci Sequence can be represented recursively by $m_n = m_{n-1}+m_{n-2}$, where m_n is the number of rabbit pairs in month n.

This sequence is powerful biologically to predict the number of rabbits after a certain time period, but it has its weaknesses. The Fibonacci sequence assumes that there is no death and no lifespan, that the rabbit pairs continue to live forever. So for this paper, the Fibonacci sequence was looked at and how lifespan changes the numerical sequences. This adds some biological validity to the sequence, although it still does not take into account random death due to diseases, overpopulation, etc.

Through the use of Mathematica programming and the Online Encyclopedia of Integer Sequences (OEIS) multiple number sequences were found and evaluated. There was no relationship found between the new Fibonacci sequences of live pairs of rabbits. But an interesting relationship was found within the death sequences of the rabbit pairs: they are Taylor polynomial expansion coefficients of $1/(1-x^2-...-x^{d-1})$, where d is the life span of the rabbit pairs.

Definition of Terms Used

Within the programming and analysis these terms were used:

- *n* is the number of months/generations that the sequences were taken out to
  - The first generation, n=1, is {1}
  - The second generation, n=2, is {1,1}
- *d* is the number of months/generations until death and *d* must be greater than or equal to 3:
  - If d = 4 a rabbit pair will be alive for four generations, will reproduce for 3 generations before dying
  - On the d^{th} generation, the rabbit pair is grandparents/not reproducing.
  - *d must be* $d \geq 3$ because if d<3. Then there will be no reproducing generations by given definition of d: the first month/generation is non-reproducing as well as the last month/generation
  - Because $d \geq 3$, the death sequence will always start with {0,0,0}, but the death sequence will always have d number of 0s to start.
- "Number Pattern" or "Number Sequence" is the sequence of rabbit pairs alive in each month/generation, as is given in the original Fibonacci sequence
- "Death Number Pattern" or "Death Sequence" is the sequence of rabbit pairs that have died in each month/generation

- "Number Alive in Each Generation" is a list of length *d* detailing the number of rabbit pairs alive in each generation
    - For example, if *d* = 4 and Number Alive in Each Generation = $\{3,2,2,1\}$, then there are 3 rabbit pairs of age one month that are still too young to reproduce, 2 rabbit pairs of age two months, 2 rabbit pairs of age three months, and 1 rabbit pair in the grandparent generation of age 4 months.

Methodology

To delve into solving this problem a Mathematica function was created to quickly evaluate multiple lifespans to create many different Number Sequences and Death Sequences out to different sizes of *n*. The Mathematica function is called FibDeathPattern with inputs of *n* and *d*:

```
FibDeathPattern [n_Integer, d_Integer] :=
 Module[{n1, n2, mylist, i, newnum, len, fiblist, fiblistnew,
   deathpattern},
  fiblist = {}; fiblistnew = {}; len = 0; deathpattern = {0, 0};
  While[len < d,
   AppendTo[fiblist, 0]
    len++
   ];
  fiblistnew = fiblist;
  n1 = 1; n2 = 1; i = 3;
  mylist = {1, 1};
  If[Abs[n] == 0, mylist = {0},
   If[Abs[n] == 1, mylist = {1}; fiblist[[1]] = 1,
    If[Abs[n] == 2, mylist = {1, 1}; fiblist[[2]] = 1;
     fiblist[[1]] = 0,
     mylist = {1, 1}; fiblist[[2]] = 1; fiblistnew = fiblist;
     While[i <= n,
      PrependTo[fiblistnew, Sum[fiblist[[r]], {r, 2, Abs[d] - 1}]];
      AppendTo[deathpattern, fiblistnew[[-1]]];
      fiblistnew = Delete[fiblistnew, -1];
      fiblist = fiblistnew;
      newnum = Sum[fiblist[[r]], {r, 1, Abs[d]}];
      AppendTo[mylist, newnum];
      i = i + 1
      ]
     ]
    ]];
  Print["Number of Generations (n) = ", n];
  Print["Life Span (d) = ", d];
  Print["Number Pattern = ", mylist];
```

```
Print["Number Alive in Each Generation = ", fiblist];
Print["Death Number Pattern = ", deathpattern]]
```

This function gives outputs of labelled *n*, *d*, Number Pattern, Number Alive in Each Generation, and Death Number Pattern. Take for example the input of FibDeathPattern [10, 4], so *n* = 10 months/generations and *d* = 4. The output given reads:

Number of Generations (n) = 10
Life Span (d) = 4
Number Pattern = {1,1,2,3,3,5,6,8,11,14}
Number Alive in Each Generation = {5,4,3,2}
Death Number Pattern = {0,0,0,0,1,0,1,1,1,2}

For the purposes of finding related number sequences, *n* was kept to 30 or above. Once a Number Pattern and Death Number Pattern were created, they were ran through OEIS.org to compare to all known sequences. Death Sequences were ran without the preceding zeros, because there were always *d* number of zeros to start.

Through this methodology, data was obtained for *d* = 3 to *d* = 8, when n *= 30*:

Lifespan (*d*)	Number Sequence	Known Sequence?	Death Sequence	Known Sequence?
3	{1,1,2,1,2,1,2,1,2,1,2,1,2,1,2,1,2,1,2,1,2,1,2,1,2,1,2,1,2,1}	Repeating {1,2}	{0,0,0,1,0,1,0,1,0,1,0,1,0,1,0,1,0,1,0,1,0,1,0,1,0,1,0,1,0,1}	Repeating {1,0}
4	{1,1,2,3,3,5,6,8,11,14,19,25,33,44,58,77,102,135,179,237,314,416,551,730,967,1281,1697,2248,2978,3945}	No	{0,0,0,0,1,0,1,1,1,2,2,3,4,5,7,9,12,16,21,28,37,49,65,86,114,151,200,265,351,465}	Padovan sequence: a(n) = a(n-2) + a(n-3) with a(0)=1, a(1)=a(2)=0 AND Coefficients of the Taylor polynomial expansion of 1/(1-x^2-x^3)
5	{1,1,2,3,5,6,10,14,21,30,45,65,96,140,206,301,442,647,949,1390,2038,2986,4377,6414,9401,13777,20192,29592,43370,63561}	No	{0,0,0,0,0,1,0,1,1,2,2,4,5,8,11,17,24,36,52,77,112,165,241,354,518,760,1113,1632,2391,3505}	Coefficients of the Taylor polynomial expansion of 1/(1-x^2-x^3-x^4)
6	{1,1,2,3,5,8,11,18,27,42,64,98,151,231,355,544,835,1281,1965,3015,4625,7096,10886,16701,25622,39308,60305,92517,141936,217752}	No	{0,0,0,0,0,0,1,0,1,1,2,3,4,7,10,16,24,37,57,87,134,205,315,483,741,1137,1744,2676,4105,6298}	Coefficients of the Taylor polynomial expansion of 1/(1-x^2-x^3-x^4-x^5)
7	{1,1,2,3,5,8,13,19,31,48,76,119,187,293,461,723,1136,1783,2800,4396,6903,10838,17018,26720,41955,65875,103434,162406,255002,400390}	No	{0,0,0,0,0,0,0,1,0,1,1,2,3,5,7,12,18,29,45,71,111,175,274,431,676,1062,1667,2618,4110,6454}	Coefficients of the Taylor polynomial expansion of 1/(1-x^2-x^3-x^4-x^5-x^6)
8	{1,1,2,3,5,8,13,21,32,52,82,131,208,331,526,836,1330,2114,3362,5345,8499,13513,21486,34163,54319,86368,137325,218348,347174,552009}	No	{0,0,0,0,0,0,0,0,1,0,1,1,2,3,5,8,12,20,31,50,79,126,200,318,506,804,1279,2033,3233,5140}	Coefficients of the Taylor polynomial expansion of 1/(1-x^2-x^3-x^4-x^5-x^6-x^7)

Conclusion

Through analysis of how lifespan affects the number sequence, no known patterns were found using OEIS.org. But, it was found that lifespan of $d = 3$ is the only number sequence that does not continually increase. Instead the number sequence is harmonic between 1 and 2 pairs of rabbits alive, and its death sequence is harmonic between 0 and 1 rabbits dying in each generation. This is the only semi-stable equilibrium. For any lifespan below $d = 3$ for how d was defined, the rabbits will die off because they will not have any reproducing generations. For any lifespan above $d = 3$ for how d was defined, both the death and number sequences are forever increasing. The harmonic nature of the sequence is most likely an artifact of the chosen definition of lifespan. Had there not been a "grandparent" generation, where the rabbits are alive but do not reproduce, at $d = 2$ the pattern would be constant 1s. Therefore, this leads to the hypothesis that at different definitions of lifespan, the number patterns will vary.

A pattern was found, though, with the Death Sequence. It was found that for $3 < d \leq 8$, the death sequences gave the coefficients of the Taylor polynomial expansion of $1/(1-x^2-...-x^{d-1})$. This pattern will most likely continue although it has to be proven.

For further in depth study, a proof of the Death Sequence pattern would be interesting, as well as adding in different coefficients of reproduction. Changing the amount that the rabbit pairs reproduce within each generation would be applicable to biology because a lot of mammals actually reproduce in larger litters. Already adding the lifespan increases the biological applicability to this numerical sequence. To be completely applicable, it would be necessary to add in carrying capacity of ecosystem so the sequences have a cap as well as adding in random disease. But both of those changes would alter the pattern to more of a random sequence.