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SOCIAL CHOICE REVERSAL PARADOXES

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by

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Approved:

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Dear Professor Housman:

Attached is one copy of the Interactive Qualifying report: Social Choice Reversal Paradoxes, Project Number DH 8801.

Sincerely,

Peter K. Harrell

Distribution: Library: 1 copy

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Abstract

This IQP project report introduces an apparently heretofore unnoted family of voting paradoxes, that might be called Social Choice Reversal Paradoxes. Such paradoxes illustrate the importance of the consideration of approval and disapproval in the comparative analysis of voting techniques. The effect of the structure of voting techniques on the resolution of Social Choice Reversal Paradoxes will be examined. Preliminary estimates of the probability of the occurrence of social choice reversal paradoxes will be presented.

Acknowledgements

I would like to thank Professor Housman for his suggestions regarding the probabilistic analysis of Social Choice Reversal Paradoxes.

Social Choice Reversal Paradoxes

The Importance of the Consideration of Approval and Disapproval In the Analysis of Voting

I

Introduction

Most paradoxes of voting have involved a choice among three or more alternatives. The conventional approach used to model voter opinion, whether a strong or weak ordering of the alternatives, has no provision for considering the matter of whether a voter disapproves of any or all of the candidates or alternatives presented as options on a ballot. There is no guarantee, that a candidate chosen in a pairwise simple majority election will in fact be acceptable to a majority of the electorate. Clearly a person may prefer A to B, yet disapprove of both. The possibility that the winner of a two candidate simple majority election might be **unacceptable** to a majority of the electorate, while a majority might find the losing opponent **acceptable** apparently has not previously been considered. The question, which one of two alternatives should be chosen as a result of a voting process, therefore can not be said to have been indisputably resolved.

A Condorcet winner is a candidate, who can defeat all opponents in pairwise simple majority elections. That is a majority of the electorate prefers the Condorcet winner to each and every opponent. In the 18th century the Marquis de Condorcet purposed the Condorcet winner as a standard for multicandidate elections. According to Condorcet, if such a candidate exists, the aggregation of individual opinion through the use of a voting technique should result in the choice of that candidate. This Condorcet standard has generally been accepted by scholars of voting as an eminently reasonable one. Just as in the two candidate case, however, there is a possibility that a Condorcet winner might be **unacceptable** to a majority of the electorate, while one of the losing opponents is **acceptable** to a majority of that same electorate. Apparently this possibility has not been considered either, when evaluating different voting techniques according to the Condorcet standard.

A Copeland winner is the candidate with the best won/lost record of all the candidates as determined from a complete set of pairwise elections results. A Condorcet winner is, therefore, always a Copeland winner, but a Copeland winner is not necessarily always a Condorcet winner. The Copeland winner was proposed as a standard by A. H. Copeland to provide a criterion for election in those instances, when no Condorcet winner exists. There is, of course, a possibility that both the Copeland winner and any victorious opponents could be **unacceptable** to some majority of the electorate, while some defeated opponent is acceptable to a majority of that same electorate. While this sort of result may not be quite so surprising as that involving a Condorcet winner, such a possibility further complicates the issue regarding what alternative should be chosen in an election process.

Finally the possibility exists, however small for elections involving more than 3 candidates, that a Condorcet loser, a candidate who can be defeated by all opponents in pairwise simple majority elections, is acceptable to some majority of the electorate, while every victorious opponent is unacceptable to a majority of that same electorate. Condorcet losers are generally thought to be among the worst of all possible choices, yet this may not always be true.

In order to demonstrate these perverse results the representation of voter preferences must include information indicating individual approval or disapproval of alternatives as well as order of preference. The examples to be presented here have been choosen both for simplicity and as evidence of what is possible. Preliminary efforts to determine the probability, that some sort of Social Choice Reversal Paradox might occur, follows this discussion of what is possible. The relative merits of resolving social choice problems according to preferential or approval/disapproval cardinal value majorities is discussed. The affect of the structure of voting techniques on voting behavior and therefore on the resolution of Social Choice Reversal Paradoxes is then examined. Finally the implications of the existence of non-quantitative individual cardinal value together with the potential occurrence of Social Choice Reversal Paradoxes for Social Choice Theory is considered.

II

Elementary Social Choice Reversal Paradox

An Elementary Social Choice Reversal Paradox is defined to exist, whenever one alternative is preferred to another by a majority of the electorate, yet that alternative is also disapproved of by a majority, while the losing opponent is approved of by a majority of that same electorate.

The simplest demonstration of the two candidate Elementary Social Choice Reversal Paradox requires only three potential voters. The first person, designated x1, prefers candidate A to candidate B, but approves of both. The second person, designated x3, again prefers candidate A to candidate B, but disapproves of both. The third person, designated y2, prefers candidate B to candidate A, approves of B but disapproves of A. These 3 people's preferences can be represented as follows.

Example 1

Preference Structure Profile

person	preference	cardinal	value
x1	A > B	a	a
x3	A > B	d	d
y2	B > A	a	d

Assuming everyone votes and votes sincerely according to preference order only, in a simple majority election between the two candidates, candidate A would receive 2 votes, candidate B only 1 vote. Candidate A would be the winner by virtue of a majority consisting of voters x1 and x3. However, candidate A is disapproved of by a majority of the electorate consisting of x3 and y2, while candidate B is approved of by a majority consisting of x1 and y2. Arguably B is the candidate that should be chosen by this small three person electorate.

Given a two candidate election and this sort of representation of individual preference structures, there are of course three other categories of personal opinion. People may have any 1 of the following 6 preference structures.

Schematic Preference Structure Profile

type	preference	cardinal	value
x1	A > B	a	a
x2	A > B	a	d
x3	A > B	d	d
y1	B > A	a	a
y2	B > A	a	d
y3	B > A	d	d

If x1, x2, x3, y1, y2 and y3 are interpreted as variables representing the number of people in each category rather than the names of individual people as before, the following 3 conditions must be met for this Elementary Social Choice Reversal Paradox to occur with candidate A the preferential majority winner and B the preferential loser.

1.1) $x1 + x2 + x3 > y1 + y2 + y3$ A preferred to B by majority
1.2) $x3 + y2 + y3 > x1 + x2 + y1$ A disapproved by a majority
1.3) $x1 + y1 + y2 > x2 + x3 + y3$ B approved by a majority

These 3 conditions can be expressed more concisely as follows.

$$(x1 + x3) - (y1 + y3) > y2 - x2 > |(x1 + y1) - (x3 + y3)|$$

Careful examination of these 2 inequalites will reveal that a delicate balance must exist between the difference in the numbers of highly "partisan" supporters of A and B, x2 and y2, and the differences both in the numbers of less "partisan" supporters of A and B, x1 + x3 and y1 + y3, and in the numbers of "optimists" and "pessimists", x1 + y1 and x3 + y3 in the electorate.

From this set of 3 necessary and sufficient conditions the following 3 necessary conditions can easily be derived and expressed as inequalities.

1.4) $x1 > y3$
1.5) $y2 > x2$
1.6) $x3 > y1$

In the first example of the Elementary Social Choice Reversal Paradox presented above x1, y2 and x3 were greater than y3, x2 and y1, respectively, by virtue of being greater than zero. Analogously the following 3 conditions must necessarily be true for the Elementary Social Choice Reversal Paradox to be possible with candidate B as the preferential majority winner.

1.7) $y1 > x3$
1.8) $x2 > y2$
1.9) $y3 > x1$

If Plurality Voting is used in a two candidate election, then preferentially sincere voting will result in the election of the preferential majority winner and the existence of an Elementary Social Choice Reversal Paradox would challenge the legitimacy of such a result. In reality given the plurality voting technique, people who disapprove of both candidates might chose to abstain, if the intensity of their dislike for one candidate is not sufficiently greater than their dislike for the other to compel them to vote. Similarly people who find both candidates acceptable might be unmotivated to vote. Should "less partisan" supporters of the preferential majority winner abstain in sufficiently greater numbers than "less partisan" supporters of the preferential loser, plurality voting might elect the preferential loser in accordance with approval and disapproval cardinal value majorities.

Actual election results depend on the behavior of potential voters. The behavior of potential voters depends on their preference structures, their knowledge both of the political circumstances and public opinion, and the voting technique employed, which governs the manner in which voters may express themselves and exercise political power. The existence of an Elementary Social Choice Reversal Paradox depends solely on the characteristics of the preference structure profile of all the potential voters regardless of voting behavior. Since society, arguably, ought to consider fully the opinions of all eligible voters, the legitimacy of the election of a preferential majority winner can be challenged on the basis of the characteristics of the preference structure profile of the entire electorate. A voting technique, that relies largely on the change occurrence of a suitably proportioned abstention of "less partisan" voters in order to resolve a Social Choice Reversal Paradox in favor of cardinal value majorities, does not adequately address that challenge.

The possible occurrence of Elementary Social Choice Reversal Paradoxes indicates that the choice between two alternatives is not a trivial social choice problem with an obvious solution based on preferential majorities. Most voting paradoxes occur only in the presence of three or more alternatives. Part of the significance of Elementary Social Choice Reversal Paradoxes is that they may occur in the presence of two or more alternatives, with any given pair of alternatives. Such paradoxes are not particularly important so long as neither candidate in the pair happens to be the candidate chosen through the application of a particular voting technique or voting standard. The following example illustrates just such an irrelevent Elementary Social Choice Reversal Paradox.

Example 2

Preference Structure Profile

number voters	Strong Order Preference	Cardinal Value	Situation Description
1	B > C > A	d d d	irrelevent Elementary
1	C > B > A	a a a	Social Choice
1	C > A > B	a a d	Reversal Paradox

C the plurality winner is approved of by a majority of the electorate. B is preferred to A by a majority, but A is approved of by a majority, while B is not, hence the existence of an irrelevent Elementary Social Choice Reversal Paradox. Since C the plurality winner is preferred to both A and B by majorities of the electorate and is approved of by a majority as well, C appears to be the only reasonable choice.

III

Condorcet Winner Social Choice Reversal Paradox

A Condorcet Winner Social Choice Reversal Paradox is defined to exist, whenever one alternative is preferred to all others in pairwise simple majority elections, yet that alternative is also disapproved of by a majority, while at least one losing opponent is approved of by a majority of that same electorate. Since in a two candidate election the preferential majority winner can be considered a Condorcet winner, the existence of an Elementary Social Choice Reversal Paradox under such circumstances is just a special two alternative case of a Condorcet Winner Social Choice Reversal Paradox. As the preceding example suggests the Condorcet Winner Social Choice Reversal Paradox can easily be demonstrated by construction of a hypothetical 3 person single peaked electorate.1

If there are M candidates on the ballot, then there are M!(M + 1) different strong order preference structure categories describing individual opinion and a set of M + 1 necessary and sufficient conditions that must be true in order for the Condorcet Winner Social Choice Reversal Paradox to occur. In a 3 candidate election, therefore there are 24 strong order preference structure categories and a set of 4 necessary and sufficient conditions (See appendix A and B for full description). Given 3 candidates A, B and C, if A is the Condorcet winner then 1) A must be preferred to B by a majority, 2) A must be preferred to C by a majority, 3) A must be disapproved of by a majority and 4) either B or C but not necessarily both must be approved of by a majority. In a 3 candidate election, if there exists a Condorcet winner and an odd number of voters with strong order preferences, then there must also exist a Condorcet loser. In this example B is the Con-dorcet loser and is disapproved of by a majority of the elector-ate. Candidate C, however, is approved of by a majority, while A the Condorcet winner is not, hence the paradox.

Example 3

Preference Structure Profile

number voters	Strong Order Preference	Cardinal Value	single peaked electorate 1 dimensional political spectrum defined by
1	A > C > B	a a d	
1	B > A > C	a d d	<--- B A C --->
1	C > A > B	a d d	<--- C A B --->

Similar three person electorates can be constructed to demonstrate the possibility of the Condorcet Winner Social Choice Reversal Paradox for multicandidate elections involving any

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1 Single peaked preferences occur when voters evaluate the alternatives in an election in terms of a single criterion, such as the liberal-conservative continuum. Suppose an individual voter's preferences are plotted with the x-axis representing a liberal conservative political spectrum, e.g. <--- L M C --->, and the y-axis indicating the degree of preference, where L denotes the liberal candidate, M the moderate and C the conservate. If that voter prefers M to L, C to L and M to C, then that voter's preferences are single peaked with respect to <--- L M C --->. However, if that voter prefers L to C, C to M and L to M, then that voter's references are not single peaked with respect to <--- L M C -->. A preference structure profile is single peaked if all voters preferences are single peaked with respect to the same criterion continuum.

number of candidates. Just as the Elementary Social Choice Reversal Paradox in a two candidate election is a special two alternative case of the Condorcet Winner Social Choice Reversal Paradox, the 6 preference structure categories describing individual opinion in the two candidate case can be considered representative of a set of 6 prototype variables, that govern the occurrence of the Condorcet Winner Social Choice Reversal Paradox in multicandidate elections (see Appendix C for 3 candidate example prototype variables). In a multicandidate election, however, there is a different set of 6 prototype variables associated with each pair of candidates on the ballot. If there are M candidates on the ballot, then there are M!/2(M - 2)! different sets of prototype variables and M!/(M - 2)! different ways a Condorcet winner can be associated with a losing opponent.

Each set of 6 prototype variables is associated with 2 sets of M + 1 necessary and sufficient conditions, one for each potential Condorcet winner in the pair. Each set of 6 prototype variables account for all M!(M + 1) strong order preference structure types, but group them differently according to the particular prototype variables in the set. As individual opinion has been represented so far, i.e. by strong orderings of the alternatives with a cardinal value of approval or disapproval associated with every alternative, each prototype variable in a set accounts for the same number of strong order preference structure variable types as any other prototype variable in the set.

From the set of 4 necessary and sufficient conditions for a 3 candidate election mentioned earlier, the following 3 necessary conditions can easily be derived and expressed in terms of the corresponding prototype variables. A is the Condorcet Winner. B is the favored losing opponent approved of by a majority of the electorate.

		Prototype Variable Descriptions	
2.5)	AB1 > BA3	AB1 --> A > B a a	BA3 --> B > A d d
2.6)	BA2 > AB2	BA2 --> B > A a d	AB2 --> A > B a d
2.7)	AB3 > BA1	AB3 --> A > B d d	BA1 --> B > A a a

Analogously if candidate B is to be the Condorcet winner and A the losing opponent, then the following 3 conditions must necessarily be true for an occurrence of Condorcet Winner Social Choice Reversal Paradox to be possible.

		Prototype Variable Descriptions	
2.8)	BA1 > AB3	BA1 --> B > A a a	AB3 --> A > B d d
2.9)	AB2 > BA2	AB2 --> A > B a d	BA2 --> B > A a d
2.10)	BA3 > AB1	BA3 --> B > A d d	AB1 --> A > B a a

The 3 different sets of prototype variables and 6 different ways a Condorcet winner can be associated with a losing opponent in a 3 candidate election can be represented according to each set of 3 necessary but insufficient conditions required for a Condorcet Winner Social Choice Reversal Paradox to exist.

1)	AB1 > BA3 BA2 > AB2 AB3 > BA1	or	BA1 > AB3 AB2 > BA2 BA3 > AB1

or

2)	BC1 > CB3 CB2 > BC2 BC3 > CB1	or	CB1 > BC3 BC2 > CB2 CB3 > BC1

or

3)	AC1 > CA3 CA2 > AC2 AC3 > CA1	or	CA1 > AC3 AC2 > CA2 CA3 > AC1

Three of the set of 4 necessary and sufficient conditions for A to be a Condorcet winner and B the favored losing opponent in an 3 candidate election can be expressed in terms of the 6 prototype variables in a manner analogous to the 2 candidate election.

$$(AB1 + AB3) - (BA1 + BA3) > BA2 - AB2 > |(AB1 + BA1) - (AB3 + BA3)$$

This same pair of inequalities applies to all multicandidate election representing 3 of the set of M + 1 necessary and sufficient condition where A is the Condorcet winner and B the favored losing opponent, long as each prototype variable is defined to represent the appropria subset of the M!(M + 1) different strong order preference structu categories.

A Condorcet winner may be involved in an Elementary Social Choi Reversal Paradox relationship with more than one opponent at a tim This sort of multiple paradox requires, that a set of M + Q necessa and sufficient conditions hold true, where Q is the number of favor losing opponents. Different 3 membered subsets of the set of M + necessary and sufficient conditions describe the relationship betwe the Condorcet winner and each favored losing opponent. All of these membered subsets share one condition in common, the Condorcet winner disapproved of by a majority of the electorate. Each 3 membered subs of conditions can be expressed in terms of an appropiate set of prototype variables arranged into a pair of inequalities in a mann analogous to the 2 candidate Elementary Social Choice Reversal Parado The following describes the relationship between candidate A, t Condorcet winner, and candidate C, the favored losing opponent.

$$(AC1 + AC3) - (CA1 + CA3) > CA2 - AC2 > |(AC1 + CA1) - (AC3 + CA3)$$

IV

Thwarted Majorities Paradox

A Thwarted Majorities Paradox exists by definition, when there exists a Condorcet winner, who loses in a multicandidate election to some opponent.2 The Marquis de Condorcet proposed the Condorcet standard, when he recognized that this sort of paradox could arise through the use of plurality voting in multicandidate elections. The possibility exists, however, that neither the Condorcet winner nor the multicandidate plurality election winner are approved of by a majority of the electorate, while some other opponent is approved of by a majority. The following example illustrates this possibility for a single peaked electorate.

Example 4

Preference Structure Profile

number voters	Strong Order Preference	Cardinal Value	Situation Description
2	A > B > C	a d d	Thwarted Majorities Paradox. Both
2	B > A > C	d d d	Plurality winner and Condorcet
3	C > B > A	a a a	winner disapproved of by a majority

single peaked electorate - 1 dimensional political spectrum defined by

<--- A B C ---> <--- C B A --->

C is both the plurality winner and Condorcet loser. B is both the Condorcet winner and Borda winner, but is disapproved of by a majority of the electorate. B is preferred by a majority to A, but A is approved of by a majority, while B is not. A is both preferred to C by a majority and approved of by a majority of the electorate, while C is not. All 3 candidates have some claim to election, although candidate C the plurality winner's claim appears to be the weakest.

Of course, the plurality winner, might be approved of by a majority, while the Condorcet winner is not. Clearly the mere existence of a Thwarted Majorities Paradox does not necessarily indicate, that the Condorcet winner is the most appropriate choice, yet estimations of the tendency to produce results consistent with the Condorcet standard are commonly used to compare different voting techniques.

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2 Peter C. Fishburn and Steven J. Brams, Paradoxes of Preferential Voting, Mathematics Magazine, vol. 56, no. 4, Sept. 1983, pg. 207.

V

Copeland Standard

The possible occurrence of the Condorcet Winner Social Choi Reversal Paradox challenges the "eminently reasonable" status of t Condorcet standard, but the significance of Elementary Social Choi Reversal Paradoxes is more fundamental. The Copeland winner w proposed as a standard to provide a criterion for election in tho instances, when no Condorcet winner exists. Elementary Social Choi Reversal Paradoxes may occur between any given pair of alternatives a are relevent to the assessment of social choice processes, whenever o alternative in the pair happens to be the alternative chosen throu the application of a particular voting technique or voting standard.

A Copeland winner may or may not be approved of by a majority the electorate. By definition, there exists an Copeland opponent th is preferred by a majority of the electorate to the Copeland winner.

The Copeland winner performs as an effective standard, howeve only in elections involving 5 or more alternatives. Although a Condo cet Winner does not always exist in elections involving only 3 or Candidates, the Copeland standard, under such circumstances, genera does not resolve the social choice problem by identifying a sing candidate as the appropriate choice. Given an electorate with stro preference orders and an odd number of voters, in a 3 candida election, either a Condorcet winner or a Cyclic Majorites Parad exists; in a 4 candidate election, in the absence of a Condorcet winn more than 1 of the 4 candidates must win 2 and lose 1. If t electorate is large, ties between 2 candidates in pairwise simp majority elections, whether resulting from an even number of voters indifferent preferences, are unlikely to occur.

Pairwise Elections - 3, 4 & 5 candidates

A vs B	A vs C	A vs D	A vs E
B vs C	B vs D	B vs E	
C vs D	C vs E		
D vs E			

A Copeland Winner Social Choice Reversal Paradox is defined exist, whenever the Copeland winner and any victorious Copeland opp nents are disapproved of by some majority of the electorate, whil some defeated Copeland opponent is approved of by a majority of th same electorate. In the case of the Copeland Winner Social Choi Reversal Paradox there is a set of M + 2 necessary and sufficie conditions that must be true, if the Paradox is to occur.

Example 5

Preference Structure Profile

number voters	Strong Order Preference	Cardinal Value				
3	A > B > C > D > E	a	d	d	d	d
2	A > B > C > D > E	a	a	d	d	d
4	B > A > C > D > E	a	a	d	d	d
1	B > C > D > A > E	a	a	a	d	d
2	C > B > D > A > E	a	a	a	d	d
3	C > D > B > E > A	a	a	d	d	d
1	D > C > E > B > A	a	d	d	d	d
2	D > E > C > B > A	a	a	d	d	d
1	E > D > C > B > A	a	a	d	d	d

19 voters

1 dimensional political spectrum defined by

<--- A B C D E --->   <--- E D C B A --->

VI

Condorcet Loser Social Choice Reversal Paradox

In a three candidate election whenever there is a Condorcet winner there is also a Condorcet loser. Previous example s.o. other than loser approved of by a majority.

If there are C candidates, then there are 2C - 1 necessary and sufficient conditions for the occurrence of the Condorcet Loser Social Choice Reversal Paradox. Unlike in the Condorcet Winner Social Choice Reversal Paradox, the Condorcet loser must have an Elementary Social Choice Reversal Paradox type relationship with every other candidate on the ballot in order for the paradox to occur. That means that each set of 6 prototype variables governing the relationship between the Condorcet loser and an opponent are relevent to any occurrence of the Condorcet Loser Social Choice Reversal Paradox. In a 3 candidate election whenever there exists a Condorcet loser there must also exist a Condorcet winner. In 3 candidate elections the Condorcet Loser Social Choice Reversal Paradox is therefore always a special case of the Condorcet Winner Social Choice Reversal Paradox.

Example 6
Preference Structure Profile

preference	cardinal value			
A > B > C > D	a	d	d	d
B > C > A > D	a	a	a	a
C > B > A > D	d	d	d	d
C > D > B > A	a	a	d	d
D > C > B > A	a	d	d	d

VII

Pseudo-Social Choice Reversal Paradoxes

The value of considering approval and disapproval in the election process is not limited to circumstances where everyone has strong order preferences. In multi-candidate elections with more than five candidates most people most likely will have weak order preferences. The fact that a person is indifferent between two candidates does not mean that society should ignore in the course of decision making whether or not that person approves or disapproves of those two candidates. Certainly such information can be useful in multi-candidate elections, but as the Elementary Social Choice Reversal Paradox suggests, may even be of value in 2 candidate elections.

Although the numbers of people who definitely prefer one of two candidates to the other determine whether or not a plurality of the entire electorate prefers one of those two candidate to the other, the number of additional people who are preferentially indifferent between those two candidates determine whether or not that plurality is in fact a majority and may also be decisive in determining whether or not some majority of the electorate approves or disapproves of one of those two candidates. Under such conditions a Pseudo-Social Choice Reversal Paradox could exist as the following example illustrates.

Pseudo-Elementary Social Choice Reversal Paradox

Preference Structure Profile

	variable & value	preference	cardinal value		
	x1 = 5	A > B	a	a	
	x2 = 3	A > B	a	d	
	x3 = 5	A > B	d	d	
*	aa = 1	A = B	a	a	*
*	dd = 2	A = B	d	d	*
	y1 = 4	B > A	a	a	
	y2 = 5	B > A	a	d	
	y3 = 2	B > A	d	d	

A vs B		A			B		
A	B		approval	13		approval	15
13	11		disapproval	14		disapproval	12

Strictly speaking A is not preferred by a majority of the entire electorate which numbers 27. The addition of 2 more voters, one with a preference structure like x1 and the other like x3 would give A a majority preferential endorsement. Whether or not this revised situation also should be called a Pseudo-Social Choice Reversal Paradox or simply a weaker version of the original is inconsequential. The fact remains that the distribution of the approval and disapproval opinions of voters with weak order preferences may challenge the legitimacy of a social choice derived from preferential considerations only.

Social choice theorists typically have chosen to ignore the communications aspect of voting and repudiate the possibility of cardinal value, focusing instead either on the exercise of political power or a depoliticized aggregation of individual preference represented by weak orderings of the alternatives. Certainly voters may actually feel, that the satisfaction derived from expressing their approval or disapproval of two candidates about whom they are indifferent exceeds the costs of voting even in a two candidate election. Merely effecting an outcome about which they are otherwise indifferent may also provide satisfaction, along with relieving what may be felt by some to be a social obligation to participate.

More importantly a clear understanding on the part of all candidates campaigning for office, of the relationship between the exercise of individual voting power, whatever that power may be, and its potential effect on the outcome, provides at least a crude and primitive basis for democratic representation. Distortion of reality is an integral facet of the practice of politics. Potential voters may not feel certain that any candidate on the ballot is satisfactory or representative of their point of view, but they can at least hope to get the attention and effect the behavior of aspiring challengers and incumbents seeking reelection. When voting techniques make it easy for candidates to hide behind the fact that their opponents are considered more vague or are more disliked than they are themselves, then the process of representation may be degraded.

Any person that understands the fundamental relationship between the exercise of political power and democratic representation, knowing that the expression of his or her preferentially indifferent cardinal opinion could affect the outcome, has at least some reason for voting. Actual representative government is a process, not simply a matter of selecting the right candidate for office. Even if a voter does not resort to semantically sincere voting to simplify a complex decision process arising from uncertain or incomplete information and multiple criteria, or have preferences that under the circumstances create no incentive to engage in semantically insincere voting, or is not in fact one of Condorcet's fabled enlightened voters who honestly attempts to judge which candidate will best serve society, a voter will almost never have anywhere near as great an incentive to conceal or misrepresent his or her opinions as candidates for public office. Presumably voters will, therefore, almost always have to form their opinions in a context of uncertainty, that may very well encourage semantically sincere voting. Ignoring people with preferentially indifferent opinions on the grounds that they have no reason to vote, much less vote sincerely, appears unjustified regardless of whether the election is a two candidate or multicandidate election.

Yes No 'Maybe So' Voting, which permits both the expression of approval and disapproval, allows voters to participate in society's decision between two candidates about whom they are preferentially indifferent. Yes No 'Maybe So' should promote competitive multicandidate elections, that fairly identify each individual candidate's actual support within the electorate as a whole. People with weak order preferences are more likely to be strongly motivated to vote in multicandidate elections, that determine final results. Depending on the particular implementation, Yes No 'Maybe So' Voting even allows society either to reject any and all candidates, who can not obtain a majority endorsement from the electorate, or if such a result is unacceptable to elect a candidate with a categorically reduced stamp of political authority, further increasing the incentive of preferentially indifferent voters to vote. Free expression of opinion and accurate representation of consensus are considered social values in and of themselves.

VIII

Cardinal Value

Social choice theorists have generally represented voter preference structures as individual orderings of the alternatives. This convention ultimately derives from the practice of Jean-Charles de Borda (1781), the Marquis de Condorcet (1785) and others involved in the early mathematical study of voting. In the 1930's the assumptions of cardinality and the interpersonal comparability of personal utilities commonly made by many economists came under severe criticism most notably from L. Robbins3. This criticism affected the subsequent development of modern social choice theory through the pioneering work of Kenneth J. Arrow, who rejected assumptions of cardinal value in the formulation and proof of his General Possibility Theorem for Social Welfare Functions, more commonly known as Arrow's Theorem.

Nevertheless, the fact remains, that people think according to both ordinal and cardinal precepts. This is clearly demonstrated by the unsurprising observation, that a voter may prefer A to B, yet still approve or disapprove of both. While a

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3 L. Robbins, An Essay on the Nature and Significance of Economic Science, Allen & Unwin, London, 1932.

strictly quantitative evaluation of the desirability of A and B is not hereby revealed, some fundamental cardinal standard, however subjective, vague, or ephemeral must be the basis of such a judgement.

Quantitative analysis of the probability of the occurrence of any one of the 3 types of Social Choice Reversal Paradoxes depends upon how voter preference structures are modelled and upon the assumptions made about the relative frequency with which voter opinions may be represented by any of the various preference structure types thereby generated. Although the existence of non-quantitative individual cardinal value judgements on the part of perspective voters is considered self-evident, there remains some ambiguity as to how these cardinal value judgements should be included in the modelling of voter preference structures. Specifically do voters always either approve or disapprove of alternatives, or might they neither approve or disapprove of some of the alternatives as well, and if so are they likely to neither approve or disapprove as frequently as they approve or disapprove.

Pondering what factors may effect the degree of resolve of individual voters as the final moment for their choosing a voting strategy approaches not only suggests that some voters who neither approve or disapprove of an alternative do so only temporarily, but evokes the multitude of criteria upon which such complex judgements are often made and the recognition that inevitably some of these criteria will be emphasized only because someone else has manipulated the political focus of the moment. Typically social choice theorists simply assume voter opinion as given. That is a legitimate approach to analysis, but when the manner in which voter opinion is represented effects both the efforts to determine the probability certain voting results will occur as well as the very implications attributed to those results, then recognition of the limitations inherent in any particular representation of voter opinion is more than merited. The cardinal value judgements attributed to perspective voters are classified as cardinal values simply because by definition they involve no comparison with other alternatives on the ballot.

When voters may neither approve or disapprove of alternatives as well as approve or disapprove, if there are M candidates on the ballot, then there are M!(3 + ((M + 4)(M - 1)/2)) different strong order preference structure categories describing individual opinion. In a 3 candidate election, therefore there are 60 strong order preference structure categories (see Appendix E) rather than 24.

IX

Competing Majorities

The question arises which majority should determine the outcome of a voting process. In the case of Social Choice Reversal Paradoxes for every pair of candidates there are three majorities to consider - the preferential majority and two approval disapproval cardinal value majorities. According to definition Social Choice Reversal Paradoxes arise in situations, where the two approval disapproval cardinal value majorities serve to mutually reinforce a contradiction of the choice normally derived from the preferential majority.

These three majorities do not consist of mutually exclusive groups of individuals. A single person will be either in the majority or the minority in two out of the three preference structure profile evaluations. Moreover these majorities have, of course, no conscious self identity. The information contained in preference structure profiles has been assumed as given for the purposes of analysis. Presumably were a Social Choice Reversal Paradox known to exist people could divide into various opposing camps according to their own preferences and individual idiosyncrasy and argue their case, but neither a conscious identity nor a common sentiment can properly be attributed to a group of people on the basis of a technical and anonymous aggregation of individual opinion. The possible occurrence of Social Choice Reversal Paradoxes should give political philosophers and social choice theorists additional reason to "question" the relationship between individual and group will.

If there were no paradox and the two approval disapproval cardinal value majorities confirmed or at least did not contradict the choice derived from the preferential majority no obvious social choice problem would exist. Just the possibility of Social Choice Reversal Paradoxes, however, raises once again questions about just what sort of information should be considered and given priority in social choice processes.

X

Conclusion - Preference or Consent

Representative democratic government claims of legitamacy are most often defended on the grounds of respect for individual rights and the consent of the governed. Examples of Social Choice Reversal Paradoxes illustrate the importance of considering approval and disapproval in social decision making processes. Unfortunately most voting techniques do not permit the voters to express dissent. If the expression of dissent is not permitted, on what grounds are claims of the existence of consent to be based?

Appendix A

In a 3 candidate election there are 24 strong order preference structure categories incorporating non-quantitative approval or diapproval cardinal value.

Approval Disapproval Strong Order Preference Structures

24 Categories - 3 Candidate Election

type	Stong Order	Cardinal Value		
xy1	A > B > C	a	a	a
xy2	A > B > C	a	a	d
xy3	A > B > C	a	d	d
xy4	A > B > C	d	d	d
xz1	A > C > B	a	a	a
xz2	A > C > B	a	a	d
xz3	A > C > B	a	d	d
xz4	A > C > B	d	d	d
yx1	B > A > C	a	a	a
yx2	B > A > C	a	a	d
yx3	B > A > C	a	d	d
yx4	B > A > C	d	d	d
yz1	B > C > A	a	a	a
yz2	B > C > A	a	a	d
yz3	B > C > A	a	d	d
yz4	B > C > A	d	d	d
zx1	C > A > B	a	a	a
zx2	C > A > B	a	a	d
zx3	C > A > B	a	d	d
zx4	C > A > B	d	d	d
zy1	C > B > A	a	a	a
zy2	C > B > A	a	a	d
zy3	C > B > A	a	d	d
zy4	C > B > A	d	d	d

Appendix B

In a 3 candidate election a set of 4 necessary and sufficient conditions must be true for the Condorcet Winner Social Choice Reversal Paradox to occur. These 4 conditions are expressed here in terms of Approval Disapproval Strong Order Preference Structures.

Necessary and Sufficient Conditions

3 candidate election

Candidate A Condorcet winner, B or C the favored Condorcet opponent

1) xy1 + xy2 + xy3 + xy4 + xz1 + xz2 +
xz3 + xz4 + zx1 + zx2 + zx3 + zx4
$>$
yx1 + yx2 + yx3 + yx4 + yz1 + yz2 +
yz3 + yz4 + zy1 + zy2 + zy3 + zy4

A preferred to B

2) xy1 + xy2 + xy3 + xy4 + xz1 + xz2 +
xz3 + xz4 + yx1 + yx2 + yx3 + yx4
$>$
yz1 + yz2 + yz3 + yz4 + zx1 + zx2 +
zx3 + zx4 + zy1 + zy2 + zy3 + zy4

A preferred to C

3) xy4 + xz4 + yx3 + yx4 + yz2 + yz3 +
yz4 + zx3 + zx4 + zy2 + zy3 + zy4
$>$
xy1 + xy2 + xy3 + xz1 + xz2 + xz3 +
yx1 + yx2 + yz1 + zx1 + zx2 + zy1

A disapproved of by a majority

4) xy1 + xy2 + xz1 + yx1 + yx2 + yx3 +
yz1 + yz2 + yz3 + zx1 + zy1 + zy2
$>$
xy3 + xy4 + xz2 + xz3 + xz4 + yz4 +
zx2 + zx3 + zx4 + zy2 + zy3 + zy4

B approved of by a majority

or

4) xy1 + xz1 + xz2 + yx1 + yz1 + yz2 +
zx1 + zx2 + zx3 + zy1 + zy2 + zy4
$>$
xy2 + xy3 + xy4 + xz3 + xz4 + yx2 +
yx3 + yx4 + yz3 + yz4 + zx4 + zy4

C approved of by a majority

Appendix C

In a multicandidate election a different set of 6 prototype variables is associated with each pair of candidates on the ballot. If there are M candidates on the ballot, then there are M!/2(M - 2)! different sets of prototype variables. Each set of 6 prototype variables account for all M!(M + 1) Approval Disapproval Strong Order Preference Structure types, but group them differently according to the particular prototype variables in the set. Therefore in a 3 candidate election there are 3 sets of prototype variables, each grouping the 24 preference structure types differently.

Prototype Variables

3 candidate election

A vs B set

AB1 = xy1 + xy2 + xz1 + zx1
AB2 = xy3 + xz2 + xz3 + zx2
AB3 = xy4 + xz4 + zx3 + zx4

BA1 = yx1 + yx2 + yz1 + zy1
BA2 = yx3 + yz2 + yz3 + zy2
BA3 = yx4 + yz4 + zy3 + zy4

B vs C set

BC1 = xy1 + yx1 + yz1 + yz2
BC2 = xy2 + yx2 + yx3 + yz3
BC3 = xy3 + xy4 + yx4 + yz4

CB1 = xz1 + zx1 + zy1 + zy2
CB2 = xz2 + zx2 + zx3 + zy3
CB3 = xz3 + xz4 + zx4 + zy4

A vs C set

AC1 = xy1 + xz1 + xz2 + yx1
AC2 = xy2 + xy3 + xz3 + yx2
AC3 = xy4 + yz4 + yx3 + yx4

CA1 = yz1 + zx1 + zx2 + zy1
CA2 = yz2 + zx3 + zy2 + zy3
CA3 = yz3 + yz4 + zx4 + zy4

Appendix D

In a 3 candidate election a set of 5 necessary and sufficient conditions must be true for a Condorcet Loser Social Choice Reversal Paradox to occur. These 5 conditions are expressed here in terms of Approval Disapproval Strong Order Preference Structures.

Necessary and Sufficient Conditions

3 candidate election

Candidate A Condorcet loser

1) yx1 + yx2 + yx3 + yx4 + yz1 + yz2 +
yz3 + yz4 + zy1 + zy2 + zy3 + zy4
\>
xy1 + xy2 + xy3 + xy4 + xz1 + xz2 +
xz3 + xz4 + zx1 + zx2 + zx3 + zx4

B preferred to A

2) yz1 + yz2 + yz3 + yz4 + zx1 + zx2 +
zx3 + zx4 + zy1 + zy2 + zy3 + zy4
\>
xy1 + xy2 + xy3 + xy4 + xz1 + xz2 +
xz3 + xz4 + yx1 + yx2 + yx3 + yx4

C preferred to A

3) xy3 + xy4 + xz2 + xz3 + xz4 + yz4 +
zx2 + zx3 + zx4 + zy2 + zy3 + zy4
\>
xy1 + xy2 + xz1 + yx1 + yx2 + yx3 +
yz1 + yz2 + yz3 + zx1 + zy1 + zy2

B disapproved of by a majority

4) xy2 + xy3 + xy4 + xz3 + xz4 + yx2 +
yx3 + yz4 + yz3 + yz4 + zx4 + zy4
\>
xy1 + xz1 + xz2 + yx1 + yz1 + yz2 +
zx1 + zx2 + zx3 + zy1 + zy2 + zy3

C disapproved of by a majority

5) xy1 + xy2 + xy3 + xz1 + xz2 + xz3 +
yx1 + yx2 + yz1 + zx1 + zx2 + zy1
\>
xy4 + xz4 + yx3 + yx4 + yz2 + yz3 +
yz4 + zx3 + zx4 + zy2 + zy3 + zy4

A approved of by a majority

Appendix E

In a 3 candidate election there are 60 strong order preference structure categories incorporating non-quantitative approval, disapproval, or neither approval or disapproval cardinal value.

Approval Disapproval Neither Stong Order Preference Structures

60 Categories - 3 Candidate Election

type	Stong Order	Cardinal Value		
uv0	A > B > C	a	a	a
uv1	A > B > C	a	a	n
uv2	A > B > C	a	a	d
uv3	A > B > C	a	n	n
uv4	A > B > C	a	n	d
uv5	A > B > C	a	d	d
uv6	A > B > C	n	n	n
uv7	A > B > C	n	n	d
uv8	A > B > C	n	d	d
uv9	A > B > C	d	d	d
uw0	A > C > B	a	a	a
uw1	A > C > B	a	a	n
uw2	A > C > B	a	a	d
uw3	A > C > B	a	n	n
uw4	A > C > B	a	n	d
uw5	A > C > B	a	d	d
uw6	A > C > B	n	n	n
uw7	A > C > B	n	n	d
uw8	A > C > B	n	d	d
uw9	A > C > B	d	d	d
vu0	B > A > C	a	a	a
vu1	B > A > C	a	a	n
vu2	B > A > C	a	a	d
vu3	B > A > C	a	n	n
vu4	B > A > C	a	n	d
vu5	B > A > C	a	d	d
vu6	B > A > C	n	n	n
vu7	B > A > C	n	n	d
vu8	B > A > C	n	d	d
vu9	B > A > C	d	d	d

continued next page

Approval Disapproval Neither Stong Order Preference Structures

type	Stong Order	Cardinal Value		
vw0	B > C > A	a	a	a
vw1	B > C > A	a	a	n
vw2	B > C > A	a	a	d
vw3	B > C > A	a	n	n
vw4	B > C > A	a	n	d
vw5	B > C > A	a	d	d
vw6	B > C > A	n	n	n
vw7	B > C > A	n	n	d
vw8	B > C > A	n	d	d
vw9	B > C > A	d	d	d
wu0	C > A > B	a	a	a
wu1	C > A > B	a	a	n
wu2	C > A > B	a	a	d
wu3	C > A > B	a	n	n
wu4	C > A > B	a	n	d
wu5	C > A > B	a	d	d
wu6	C > A > B	n	n	n
wu7	C > A > B	n	n	d
wu8	C > A > B	n	d	d
wu9	C > A > B	d	d	d
wv0	C > B > A	a	a	a
wv1	C > B > A	a	a	n
wv2	C > B > A	a	a	d
wv3	C > B > A	a	n	n
wv4	C > B > A	a	n	d
wv5	C > B > A	a	d	d
wv6	C > B > A	n	n	n
wv7	C > B > A	n	n	d
wv8	C > B > A	n	d	d
wv9	C > B > A	d	d	d

Appendix F

In a 3 candidate election a set of 4 necessary and sufficient conditions must be true for the Condorcet Winner Social Choice Reversal Paradox to occur. These 4 conditions are expressed here in terms of Approval Disapproval Neither Strong Order Preference Structures.

Candidate A Condorcet winner, B or C the favored Condorcet opponent

Candidate A is preferred to Candidate B

1) uv0 + uv1 + uv2 + uv3 + uv4 + uv5 + uv6 + uv7 + uv8 + uv9 +
uw0 + uw1 + uw2 + uw3 + uw4 + uw5 + uw6 + uw7 + uw8 + uw9 +
wu0 + wu1 + wu2 + wu3 + wu4 + wu5 + wu6 + wu7 + wu8 + wu9
>
vu0 + vu1 + vu2 + vu3 + vu4 + vu5 + vu6 + vu7 + vu8 + vu9 +
vw0 + vw1 + vw2 + vw3 + vw4 + vw5 + vw6 + vw7 + vw8 + vw9 +
wv0 + wv1 + wv2 + wv3 + wv4 + wv5 + wv6 + wv7 + wv8 + wv9

Candidate A is preferred to Candidate C

2) uv0 + uv1 + uv2 + uv3 + uv4 + uv5 + uv6 + uv7 + uv8 + uv9 +
uw0 + uw1 + uw2 + uw3 + uw4 + uw5 + uw6 + uw7 + uw8 + uw9 +
vu0 + vu1 + vu2 + vu3 + vu4 + vu5 + vu6 + vu7 + vu8 + vu9
>
wu0 + wu1 + wu2 + wu3 + wu4 + wu5 + wu6 + wu7 + wu8 + wu9 +
wv0 + wv1 + wv2 + wv3 + wv4 + wv5 + wv6 + wv7 + wv8 + wv9 +
vw0 + vw1 + vw2 + vw3 + vw4 + vw5 + vw6 + vw7 + vw8 + vw9

Candidate A is disapproved of by a majority

3) uv9 + uw9 +
vu5 + vu8 + vu9 + wu5 + wu8 + wu9 +
vw2 + vw4 + vw5 + vw7 + vw8 + vw9 +
wv2 + wv4 + wv5 + wv7 + wv8 + wv9
>
uv0 + uv1 + uv2 + uv3 + uv4 + uv5 + uv6 + uv7 + uv8 +
uw0 + uw1 + uw2 + uw3 + uw4 + uw5 + uw6 + uw7 + uw8 +
vu0 + vu1 + vu2 + vu3 + vu4 + vu6 + vu7 +
wu0 + wu1 + wu2 + wu3 + wu4 + wu6 + wu7 +
vw0 + vw1 + vw3 + vw6 +
wv0 + wv1 + wv3 + wv6

Candidate B is not disapproved of by a majority

4) uv0 + uv1 + uv2 + uv3 + uv4 + uv6 + uv7 +
uw0 + uw1 + uw3 + uw6 +
vu0 + vu1 + vu2 + vu3 + vu4 + vu5 + vu6 + vu7 + vu8 +
vw0 + vw1 + vw2 + vw3 + vw4 + vw5 + vw6 + vw7 + vw8 +
wu0 + wu1 + wu3 + wu6 +
wv0 + wv1 + wv2 + wv3 + wv4 + wv6 + wv7
>
uv5 + uv8 + uv9 +
uw2 + uw4 + uw5 + uw7 + uw8 + uw9 +
vu9 + vw9 +
wu2 + wu4 + wu5 + wu7 + wu8 + wu9 +
wv5 + wv8 + wv9

or

Candidate B is approved of by a majority

4) uv0 + uv1 + uv2 + uw0+
vu0 + vu1 + vu2 + vu3 + vu4 + vu5 +
vw0 + vw1 + vw2 + vw3 + vw4 + vw5 +
wu0 + wv0 + wv1 + wv2
>
uv3 + uv4 + uv5 + uv6 + uv7 + uv8 + uv9 +
uw1 + uw2 + uw3 + uw4 + uw5 + uw6 + uw7 + uw8 + uw9 +
vu6 + vu7 + vu8 + vu9 +
vw6 + vw7 + vw8 + vw9 +
wu1 + wu2 + wu3 + wu4 + wu5 + wu6 + wu7 + wu8 + wu9 +
wv3 + wv4 + wv5 + wv6 + wv7 + wv8 + wv9

or

Candidate C is not disapproved of by a majority

4) uw0 + uw1 + uw2 + uw3 + uw4 + uw6 + uw7 +
uv0 + uv1 + uv3 + uv6 +
wu0 + wu1 + wu2 + wu3 + wu4 + wu5 + wu6 + wu7 + wu8 +
wv0 + wv1 + wv2 + wv3 + wv4 + wv5 + wv6 + wv7 + wv8 +
vu0 + vu1 + vu3 + vu6 +
vw0 + vw1 + vw2 + vw3 + vw4 + vw6 + vw7
>
uw5 + uw8 + uw9 +
uv2 + uv4 + uv5 + uv7 + uv8 + uv9 +
wu9 + wv9 +
vu2 + vu4 + vu5 + vu7 + vu8 + vu9 +
vw5 + vw8 + vw9

or

Candidate C is approved of by a majority

4) uw0 + uw1 + uw2 + uv0 +
wu0 + wu1 + wu2 + wu3 + wu4 + wu5 +
wv0 + wv1 + wv2 + wv3 + wv4 + wv5 +
vu0 + vw0 + vw1 + vw2

$>$

uw3 + uw4 + uw5 + uw6 + uw7 + uw8 + uw9 +
uv1 + uv2 + uv3 + uv4 + uv5 + uv6 + uv7 + uv8 + uv9 +
wu6 + wu7 + wu8 + wu9 +
wv6 + wv7 + wv8 + wv9 +
vu1 + vu2 + vu3 + vu4 + vu5 + vu6 + vu7 + vu8 + vu9 +
vw3 + vw4 + vw5 + vw6 + vw7 + vw8 + vw9

Appendix G

In a multicandidate election a different set of 6 prototype variables is associated with each pair of candidates on the ballot. If there are M candidates on the ballot, then there are M!/2(M - 2)! different sets of prototype variables. Each set of 6 prototype variables account for all M!(M + 1) Approval Disapproval Neither Strong Order Preference Structure types, but group them differently according to the particular prototype variables in the set. Therefore in a 3 candidate election there are 3 sets of prototype variables, each grouping the 24 preference structure types differently.

Prototype Variables

3 candidate election

A vs B set

AB1 = uv0 + uv1 + uv2 + uv3 + uv4 + uv6 + uv7 + uw0 + uw1 + uw3 + uw6 + wu0 + wu1 + wu3 + wu6

AB2 = uv5 + uv8 + uw2 + uw4 + uw5 + uw7 + uw8 + wu2 + wu4 + wu7

AB3 = uv9 + uw9 + wu5 + wu8 + wu9

BA1 = vu0 + vu1 + vu2 + vu3 + vu4 + vu6 + vu7 + vw0 + vw1 + vw3 + vw6 + wv0 + wv1 + wv3 + wv6

BA2 = vu5 + vu8 + vw2 + vw4 + vw5 + vw7 + vw8 + wv2 + wv4 + wv7

BA3 = vu9 + vw9 + wv5 + wv8 + wv9

A vs C set

AC1 = uv0 + uv1 + uv3 + uv6 + vu0 + vu1 + vu3 + vu6 + uw0 + uw1 + wu2 + uw3 + uw4 + uw6 + uw7

AC2 = uw5 + uw8 + uv2 + uv4 + uv5 + uv7 + uv8 + vu2 + vu4 + vu7

AC3 = uv9 + uw9 + wu5 + wu8 + wu9

CA1 = vw0 + vw1 + vw3 + vw6 + wv0 + wv1 + wv3 + wv6 + wu0 + wu1 + wu2 + wu3 + wu4 + wu6 + wu7

CA2 = wu5 + wu8 + wv2 + wv4 + wv5 + wv7 + wv8 + vw2 + vw4 + vw7

CA3 = wu9 + wv9 + vw5 + vw8 + vw9

B vs C set

BC1 = uv0 + uv1 + uv3 + uv6 + vu0 + vu1 + vu3 + vu6 +
vw0 + vw1 + vw2 + vw3 + vw4 + vw6 + vw7
BC2 = vw5 + vw8 + vu2 + vu4 + vu5 + vu7 + vu8 +
uv2 + uv4 + uv7
BC3 = vw9 + vu9 + uv5 + uv8 + uv9

CB1 = uw0 + uw1 + uw3 + uw6 + vu0 + vu1 + vu3 + vu6 +
wv0 + wv1 + wv2 + wv3 + wv4 + wv6 + wv7
CB2 = wv5 + wv8 + wu2 + wu4 + wu5 + wu7 + wu8 +
uw2 + uw4 + uw7
CB3 = wv9 + wu9 + uw5 + uw8 + uw9