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THE NUCLEOLUS AND SHAPLEY VALUE FOR COOPERATIVE MATCHING GAMES ON WEIGHTED GRAPHS

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Abstract: In any cooperative game we are often concerned with fairly allocating the savings to the players in the game. This paper introduces a specific cooperative game, called the matching game, and a procedure for finding the nucleolus and Shapley value (two fair allocations) for different classes of matching games.

Introduction: An n-person cooperative game is a pair (N,w), where $N = \{1,2,3,\ldots,n\}$ is the set of players and $w(S)$ is a real-valued function from the subsets of N to the real numbers which satisfies $w(\emptyset)=0$. The subsets of N can be thought of as coalitions of players in N, and w can be interpreted as the worth function because $w(S)$ represents the relative worth of the coalition S. A cooperative game is called *superadditive* if $w(S \cup T) \geq w(S) + w(T)$ for all $S,T \in N$ satisfying $S \cap T = \emptyset$. More plainly, superadditivity ensures that when two or more disjoint players or coalitions decide to cooperate their worth will be greater than or equal to the sum of their individual worths.

Many cooperative games can be defined with respect to graphs. A *graph* is a finite nonempty set of objects, called *vertices*, together with a possibly empty set of unordered pairs of distinct vertices called *edges*. In a graph-restricted game the elements of N are the vertices, and an edge represents a coalition between two players. A *weighted graph* is a graph in which each edge e is assigned a positive real number, called the *weight* of e. In a weighted graph game the weight on each edge corresponds to the worth of a two player coalition, the specific coalition determined by the end vertices of that edge.

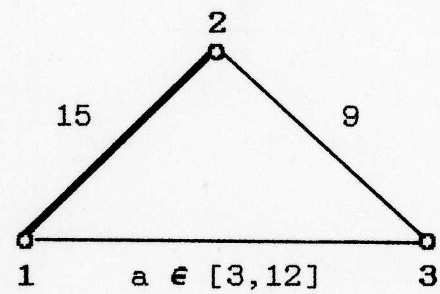
An *induced subgraph* of a graph G contains a nonempty subset of the vertices of G, call it U, and all edges of G both of whose end vertices are contained in U.

Two distinct edges in a graph are called *independent* if they are not adjacent (i.e. if they do not share any vertices). A *matching* is a set of independent edges. In a weighted graph, a *maximum weight matching* is one such that the sum of the weights of the edges contained in the matching is maximized. The class of graph games that will be discussed in this paper are called *matching games*. A *matching game* on a weighted graph is defined by N players which can form coalitions of worth $w(S)$. The worth of a coalition is the maximum weight matching on the subgraph induced by the players in that coalition. Edmonds and Johnson (1970) introduced an algorithm for finding the maximum weight matching on any graph.

The superadditivity of matching games may not be intuitively obvious. Consider the disjoint subsets S and T of N. The worth of each coalition is equal to the maximum weight matching on that subset, which is always ≥ 0. In combining these two subsets the maximum weight matching of each could be preserved, in which case

$w(S \cup T) = w(S) + w(T)$. The only basis for breaking up the maximum weight matchings of each subset would be if it were possible to find a higher weight matching in the graph $S \cup T$. Therefore, matching games are superadditive. An example of a matching game is illustrated below. In is easy to see that this simple game is superadditive.

EXAMPLE: $w(\{1,2,3\})=15$
$w(\{1,2\})=15$
$w(\{1,3\})=a$
$w(\{2,3\})=9$
$w(\{1\})=w(\{2\})=$
$w(\{3\})=0$



A particular matching game called the *Assignment game* was described by Shapley and Shubik (1972) and others and involves two finite disjoint sets of players, S and T. Associated with each possible partnership (i,j), is a real number which represents the relative value of that coalition. The only coalitions of any worth are those that contain players from each set. The assignment game can be represented as a bipartite graph, and is easily applicable to a market model. One set of players could represent buyers, and the other sellers. The goal is to find a maximum weight matching on the graph, which would correspond to making the most favorable matches between buyers and sellers.

When discussing cooperative games we are often concerned with the allocation, or payoff that each player receives. An allocation $x \in \mathbb{R}^n$ is *efficient* for the game *(N,w)* if

$$\sum_{i \in N} x_i = w(N)$$

It makes sense that the sum of the allocations to each individual is equal to the worth of the grand coalition. We would expect any fair allocation to have this property.

Another property that would be indicative of a fair allocation is called *individual rationality*. An allocation is individually rational if the payoff to each individual *i* is at least *w({i})*. The set of all individually rational and efficient allocations is called the set of *imputations*.

The *core* of a game *w* is the set of all *n*-vectors *x* satisfying:

$$\sum_{i \in S} x_i \geq w(S),\ S \subset N,$$
$$\sum_{i \in N} x_i = w(N).$$

In other words, the core is the set of all group rational and efficient allocations. An allocation is *group rational* if the sum of the payoff to any coalition *(S)* is greater than or equal to *w(S)*. The core of the above game is:

$$\begin{gathered} x_1+x_2+x_3=15 \\ x_1+x_2 \geq 15 \\ x_1+x_3 \geq a \\ x_2+x_3 \geq 9 \\ x_1, x_2, x_3 \geq 0 \end{gathered}$$

Note that when a = 3 the core is nonempty, but when a = 12 a problem arises. There is no solution to the above system of equations, and therefore the core must be empty.

The *Shapley value* is an allocation method developed by Lloyd Shapley which allocates to each particular player its average marginal contribution over all possible permutations of the players. The Shapley value for individual i in closed form is:

$$\Phi_i(N,w)=\sum_{S\subseteq N}\frac{(s-1)!(n-s)!}{n!}[w(S)-w(S-(i))]$$

where s denotes $|S|$. In the above example the Shapley value produces the allocation (5,8,2) for players 1,2 and 3 respectively when a = 3. When a = 12, the allocation is (6.5,5,3.5). The sum of the values is 15 in either case, showing that this is an efficient allocation. The Shapley value is not necessarily group rational, and therefore may not be an element of the core, as it is not in this example. Computing the Shapley value can be tedious, especially when the game involves many players.

Given a cooperative game (N,w), let

$$e(x,S)=v(S)-\sum_{i\in S}x_i$$

be the *complaint* that coalition S has against the allocation x_i. $e(x,S)$ is the difference between what group S can obtain on its own and what it will receive according to the allocation x. The *nucleolus* is the imputation which lexicographically minimizes the maximum complaints (i.e. the largest complaint is as small as possible and is voiced by a few groups as possible, the next largest complaint is as small as possible and is voiced by as few groups as possible, etc.). The nucleolus is group rational, therefore it is always an element of the core if the core is nonempty. The nucleolus for the example is (4.5,10.5,0) for players 1,2 and 3 respectively when a = 3. When a = 12, the nucleolus is (8,5,2). Note that this allocation is not an element of the core and therefore, as we noted before, the core must be empty. A linear programming method can be used to find the nucleolus.

A theorem by E. Kohlberg characterizes the nucleolus in a distinctive way. For a given allocation $x \in X$ (where X is the set of all imputations) define a sequence of collections $\beta_1, \beta_2, \ldots, \beta_q$, such that β_1 is the collection of coalitions with the greatest complaint (maximum $e(x,S)$). β_2 would be the collection of coalitions with the next largest complaint, and so on such that every $S \subset N$ belongs to one of the β_k. Define also for $k = 1,2,\ldots,q$,

$$\zeta_k = \bigcup_{j=1}^{k} \beta_j$$

A paraphrase of Kohlberg's theorem:

Theorem: A necessary and sufficient condition for $x \in X$ to be the nucleolus of the superadditive game *(N,w)* is that, for every $k = 1,2,...q$, the collections ζ_k determined by x are balanced.

A set of coalitions is *balanced* if there exists coefficients λ_s satisfying

$$\sum_{\substack{S\in\zeta_k \\ i\in S}} \lambda_S = 1, \; i\in N$$
$$\lambda_S > 0, \; S\in\zeta_k$$

For example, given the set $\beta_1 = \zeta_1 = \{\{1\},\{2,3\},\{2,4\},\{3,4\}\}$, the values $\lambda_1=1$, $\lambda_{23}=\lambda_{24}=\lambda_{34}=\frac{1}{2}$ could be assigned to each of the four coalitions respectively to make β_1 balanced, as is shown below.

$$\sum_{\substack{S\in\zeta_1 \\ 1\in S}} \lambda_S = \lambda_1 = 1$$
$$\sum_{\substack{S\in\zeta_1 \\ 2\in S}} \lambda_S = \lambda_{23} + \lambda_{24} = \tfrac{1}{2} + \tfrac{1}{2} = 1$$
$$\sum_{\substack{S\in\zeta_1 \\ 3\in S}} \lambda_S = \lambda_{23} + \lambda_{34} = \tfrac{1}{2} + \tfrac{1}{2} = 1$$
$$\sum_{\substack{S\in\zeta_1 \\ 4\in S}} \lambda_S = \lambda_{24} + \lambda_{34} = \tfrac{1}{2} + \tfrac{1}{2} = 1$$

The set of values that make a given collection balanced is called the *balancing vector*.

Another theorem concerning balanced collections will also prove useful:

Theorem: The union of balanced collections is balanced.

Proof: Let

$$C = \{S_1, \ldots, S_m\},$$

$$D = \{T_1, \ldots, T_k\}$$

be balanced collections, with balancing vectors $(y_1, \ldots, y_m)$ and $(z_1, \ldots, z_k)$ respectively. Then the union of these two collections is

$$C \cup D = \{R_1, \ldots, R_q\},$$

where $q \leq m+k$, since there may be some overlap in the collections. For any t, $0 < t < 1$, define

$$w_j = \begin{cases} ty_1 & \text{if } R_j = S_1 \in C-D \\ (1-t)z_p & \text{if } R_j = T_p \in D-C \\ ty_1 + (1-t)z_p & \text{if } R_j = S_1 = T_p \in C \cap D \end{cases}$$

then $(w_1, \ldots, w_q)$ is a balancing vector for $C \cup D$. Therefore the union of two balanced collections is balanced, and by induction the union of any number of balanced collections is balanced.

It is easier to grasp the proof of this theorem using an example. Consider the two balanced sets below:

$$C = \{\{1,2,3\},\{2,3\},\{1\}\}$$

$$D = \{\{1,2\},\{2,3\},\{1,3\}\}$$

the balancing vectors for each set are $(\frac{1}{2},\frac{1}{2},\frac{1}{2})$ and $(\frac{1}{2},\frac{1}{2},\frac{1}{2})$, respectively. The union of the two sets is:

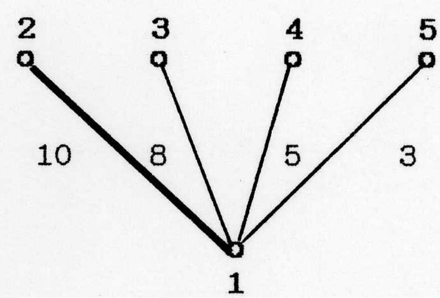
$$C \cup D = \{\{1,2,3\},\{2,3\},\{1\},\{1,2\},\{1,3\}\}$$

Using the formula from the proof, we now construct a balancing vector for this new set. The vector turns out to be $(\frac{1}{2}t, \frac{1}{2}, (\frac{1}{2}-\frac{1}{2}t), (\frac{1}{2}-\frac{1}{2}t), \frac{1}{2}t)$. Summing over each player, we find that each sum is 1, and therefore the set is balanced.

Now we will try and use this information in finding the nucleolus and Shapley value for matching games on specific classes of graphs.

Stars. A graph G is *bipartite* if it is possible to partition its vertex set V(G) into 2 subsets V_1, V_2 such that every element of the edge set E(G) joins a vertex of V_1 to a vertex of V_2. (i.e. there are no edges between vertices in the same subset). A *star* is a very simple bipartite graph in which the size of one subset is 1 ($|V_2| = 1$). The maximum weight matching in any star is equal to the maximum weight edge.

EXAMPLE: $w(\{1,2,3,4,5\})=w(\{1,2\})=10$
nucleolus = (9,1,0,0,0)



Given any star, the edges can be arranged so that they are in nonincreasing order from left to right. Therefore, we can always refer to (1,2) as the highest weight edge, and (1,3) as the next highest, and so on.

Proposition: The nucleolus for any n-person matching game on a star is:

$$[(\tfrac{1}{2}(w(\{1,2\})+w(\{1,3\})),(\tfrac{1}{2}(w(\{1,2\})-w(\{1,3\})),0,0,\ldots,0].$$

Proof: Assume that the imputation above is the nucleolus. Consider the complaints for each coalition of players:

S	Complaint
$\{1,2,3,4,...,n\}$	0
$\{1,2,...\}$ any coalition with 12	0
$\{1,\ 3,...\}$ any coalition with 13 and not 2	$w(\{1,3\}) - \frac{1}{2}(w(\{1,2\})+w(\{1,3\}))= \frac{1}{2}(w(\{1,3\})-w(\{1,2\}))$
$\{1,\ \ 4,...\}$ any coalition with 14 and not 23	$w(\{1,4\}) - \frac{1}{2}(w(\{1,2\})+w(\{1,3\}))$
$\{1,\ \ \ 5,...\}$, $\{1,\ \ \ \ 6,...\},...\{1\ \ n\}$	etc.
$\{\ 2,...\}$ any coalition with 2 and not 1	$0 - \frac{1}{2}(w(\{1,2\})-w(\{1,3\})) = \frac{1}{2}(w(\{1,3\}) - w(\{1,2\}))$
$\{\ \ 3,...\}$ any coalition not containing 1 or 2	0
$\{1\}$	$0 - \frac{1}{2}(w(\{1,2\}) + w(\{1,3\}))$

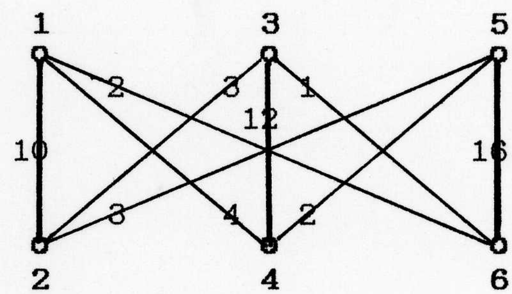
We can now order these complaints according to the β_k's in Kohlberg's theorem.

$$\beta_1 = \{\{1,2,3,4,..,n\},\{1,2,..\},\{\ \ 3,...\},\{\ \ \ 4,...\},...,\{n\}\}$$
$$\beta_2 = \{\{1,\ 3,...\},\{\ 2,...\}\}$$
$$\beta_3 = \{\{1,\ \ 4,..\}\}$$
$$\beta_4 = \{\{1,\ \ \ 5,..\}\}$$
$$\vdots$$
$$\beta_i = \{\{1\}\}$$

Now if we consider $\zeta_1 = \beta_1$, $\zeta_2 = \beta_1 \cup \beta_2$, etc. we will find that these sets are balanced, and therefore the conjectured imputation must be the nucleolus.

Other Bipartite graphs. Now we will consider a more complex class of bipartite graphs, *K(n,n)* graphs. The symbol *K(n,n)* stands for a complete bipartite graph with equal numbers of vertices in each subset. A *complete* bipartite graph is a bipartite graph with vertex sets V_1 and V_2 with the added property that if $u \in V_1$ and $v \in V_2$ then $uv \in E(G)$ (where E(G) is the edge set of G). A K(3,3) graph is pictured below:

Example: nuc. = (5,5,6,6,8,8)



Given a complete bipartite graph, the vertices can be relabeled such that the edges in the maximum weight matching have end vertices 1 and 2, 3 and 4, 5 and 6, etc. respectively. Using this labeling we can now use the following formula to find the nucleolus.

Proposition: Given a complete bipartite graph K(n,n), if the weight of each edge in the maximum weight matching is greater than or equal to 2x each edge <u>not</u> in the maximum matching then the nucleolus is:

$$(\tfrac{1}{2}w(\{1,2\}),\tfrac{1}{2}w(\{1,2\}),\tfrac{1}{2}w(\{3,4\}),\tfrac{1}{2}w(\{3,4\}),\ldots,\tfrac{1}{2}w(\{m,n\}),\tfrac{1}{2}w(\{m,n\}))$$

Proof: Assume we are given K(n,n) with each edge in the maximum weight matching greater than each edge not in the maximum weight matching. Consider the complaints of some of the coalitions.

The following coalitions have a complaint of 0, and therefore would be in β_1:

$$\{\text{grand coalition}\},$$

$$\{1,2\},\{3,4\},\ldots,\{m,n\} \quad \text{and any union of these pairs}$$

It is not so obvious what the elements of β_2 should be, it depends on the actual weights on the edges. Assume that

$$\beta_2 = \{\{1\},\{2\}\}$$

which is an option since $\beta_1 \cup \beta_2$ is balanced. This means that $\{1\}$ and $\{2\}$ have equal complaints, $-x_1 = -x_2$. Since we know that $x_1 + x_2 = w(\{1,2\})$ then by substitution we arrive at

$$x_1 = x_2 = \tfrac{1}{2}w(\{1,2\}).$$

Assume then that $\beta_3 = \{\{3\},\{4\}\}$, and $\beta_4 = \{\{5\},\{6\}\}$, etc. Through the same method as above we find that:

$$x_3 = x_4 = \tfrac{1}{2}w(\{3,4\}), \text{ and } \quad x_5 = x_6 = \tfrac{1}{2}w(\{5,6\}), \text{ etc.}$$

By making these assumptions we form a number of different inequalities, because by definition these complaints must be greater than the complaints of all of the remaining coalitions. For example, the complaint for player 1 must be greater than the complaint of the coalition $\{2,3\}$

$$-\tfrac{1}{2}w(\{1,2\}) > w(\{2,3\}) - \tfrac{1}{2}w(\{1,2\}) - \tfrac{1}{2}w(\{3,4\})$$

or simply $$w(\{3,4\}) > 2w(\{2,3\})$$

Looking at a number of these equations yields the following result:

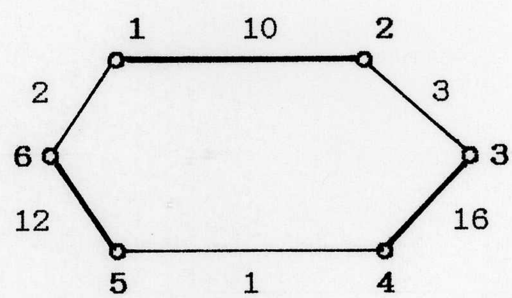
$$w(\{1,2\}),w(\{3,4\}),\ldots w(\{m,n\}) > 2(\text{any edge not in the matching})$$

Which shows that, when the edges in the maximum weight matching are greater than 2x the weights of the edges not in the matching, then the nucleolus allocates to each player exactly ½ the weight of the matching edge incident to that player.

In the example above all of the assumptions are satisfied, and the nucleolus does in fact allocate the conjectured amount to each player.

This proof can also be used when considering *even cycles* and *incomplete bipartite graphs* with equal numbers of vertices in each subset. A *cycle* is an alternating series of vertices and edges, beginning and ending with the same vertex, in which no vertex or edge is repeated (other than the beginning vertex).

Example: nuc. = (5,5,8,8,6,6)



An even cycle, like the one above, has an even number of vertices and is just a special bipartite graph. An incomplete bipartite graph is a K(n,n) with some edges removed. Both of these graphs can be transformed into complete bipartite graphs by adding 0-weight "dummy" edges, then the proposition applies.

Conclusion. These two examples prove that it can be fairly simple to find the nucleolus of different classes of matching games. The result dealing with even cycles leads me to believe that a similar result exists also for odd cycles, and that the result for odd cycles may be related to complete bipartite graphs with vertex subsets of different sizes. I have more results on 4-cycles (cycles with 4 vertices) for cases where the split between players is not 50/50. However, when the game involves more than four players it is much more complicated to find the allocations to each player.

Simple methods for finding the Shapley value have proven much more elusive because there is no characterization of the Shapley value like Kohlberg´s characterization of the nucleolus. In the star graph I have found some results, but they are not easily generalized. The bipartite case is even more complex.

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