The Shapley Value on Partially Defined Games

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Normally, a cooperative game is defined as a pair (N,v) where $N = \{1,2,...,n\}$ and v is a real valued function on the subsets of N. But, this definition assumes that we know a large amount of information about the game. The problem I'm dealing with is how can you find an allocation if you don't know the value of v on all of the 2^n subset of N.

To deal with this, I have defined *partially defined game*(PDG) as a triple (N,Z,v) where Z is a collection of subsets of the set of players N and v is a real valued function on Z. Also, we add the restriction that Z must contain both the grand coalition N and each individual player in N. In this paper we will only be dealing with 0-monotonic partially defined games, which says that if v is the 0 normalization of your PDG then $v(\{i\}) = 0$ for all $i \in N$ and $v(S) \geq v(T)$ for all $S,T \in Z$ with $T \subseteq S$. We will also be working with a class of PDGs referred to as symmetric meaning that we have a subset of indices $J \subseteq N$ and $Z = \{ S \subseteq N \mid |S| \in J \}$. In other works the structure of Z is unchanging under permutations of the set of players.

Under a normal cooperative game the Shapley value can be characterized as the unique allocation method that satisfies the efficiency, additivity, symmetry and dummy properties, or by Chun's characterization the unique value on which efficiency, triviality, coalitional strategic equivalence and fair ranking hold. Unfortunately, the definitions of these axioms don't work entirely in their present form, so I have modified the definitions as follows:

An allocation ϕ is called *efficient* if for the PDG (Z,v) ~~if~~ $\sum \phi_i(v) = v(N)$.

An allocation ϕ is called *additive* if for all PDG u, v, and w on the same set of subsets Z such that $w(S) = u(S) + v(S)$ for all $S \in Z$ then $\phi(w) = \phi(u) + \phi(v)$.

An allocation ϕ is called *symmetric* on PDG if for all permutations π of N and all individuals $i \in N$, it follows that $\phi_{\pi(i)}(Z,\pi v) = \phi_i(Z,v)$ where πv is the worth function defined by $\pi v(\pi S) = v(S)$ for all $S \in Z$.

An allocation ϕ satisfies the *dummy player property* if for all PDG (N,Z,v) and all dummy players $i \in N$ then $\phi_i = 0$. Where player i is dummy if for all 0-monotonic covers v' of v, player i is dummy in v'. define this

define 0-monoton cover

An allocation ϕ satisfies *triviality* if $\phi(v_0) = 0$, where $v_0(S) = 0$ for all $S \in Z$.

An allocation ϕ satisfies *coalitional strategic equivalence* if for all $T \subseteq N$ such that $T \neq \emptyset$, and for all $\alpha \in \mathbb{R}$, if $v = w + w_{\alpha,T}$, then $\phi_i(v) = \phi_i(w)$ for all $i \in N\setminus T$. Where $w_{\alpha,T}(S) = \alpha$ if ~~$S \subseteq T$~~ $S \in Z$ and $T \subseteq S$ and $= 0$ otherwise.

An allocation ϕ satisfies *fair ranking* in a PDG if for all $T \subseteq N$, if $v(S) = w(S)$ for all $S \in Z$ with $S \neq T$, then $\phi(v) > \phi_j(v)$ implies $\phi_i(w) > \phi_j(w)$ for all $i,j \in T$.

An allocation ϕ satisfies *marginality* if for all $i \in N$, ← How is this defined when S, $S \cup i$, $S-i$ are not in Z? if $\Delta_i v = \Delta_i w$, then $\phi_i(v) = \phi_i(w)$. Where $\Delta_i v = v(S) - v(S\setminus i)$ if $i \in S$, and $\phi_i v = v(S \cup i) - v(S)$ if $i \notin S$.

An allocation ϕ is *individually rational* $\phi_i(v) \geq v(\{i\})$.

An allocation ϕ is *individually reasonable* $\phi_i(v) \leq \max\{ v(S) - v(S-\{i\}) \mid i \in S \in Z\}$.

The problem now is to determine ~~is~~ if an extension of the Shapley value exists, or if it only exists in certain cases. Using these definitions, I have found a unique value on 0-monotonic symmetric partially defined games that holds under efficiency, additivity, symmetry and dummy on partially defined games, defined as follows:

you really need additivity and proportionality which together are called linearity

$$\phi_i(v) = \sum_{i\in S\in Z} \frac{(n-s)!(s-1)!}{n!}\, v(S) \; - \sum_{i\notin S\in Z} \frac{(n-s-1)!s!}{n!}\, v(S)$$

This particular value also satisfies the other axioms listed above.

<u>Theorem 1:</u> ϕ is the unique allocation method on 0-monotonic PDG's that satisfies efficiency, additivity, symmetry and dummy.

<u>Proof:</u> We will start with a value which is only additive and develop it into the value we want.

Since ϕ is additive, we can write it in the form:

$$\phi_i(v) = \sum_{S\in Z} c_i(S)v(S)$$

Using symmetry and efficiency, we can show that

$$\text{where } c_i(S) = C_{|s|} / |S| \qquad \text{if } i \in S$$
$$= -C_{|s|} / (n - |S|) \qquad \text{if } i \notin S$$

Also, by efficiency $C_n = 1$. Finally we must calculate the set of constants C_i using dummy. We will consider the following class of 0-monotonic simple PDGs v_{ik}. Where $v_{ik}(S) = 1$ if $|S| > k$ or $|S| = k$ and $i \notin S$, and $v_{ik}(S) = 0$ otherwise. In each of these games player i is a dummy player so $\phi_i(v) = 0$.

true, but another sentence or two of explanation would be helpful

actually this is not true for $k=1$; V_{i1} is not 0-monotonic

$$\phi_i(v) = \sum_{i\in S\in Z} \frac{C_s v_{ik}(S)}{s} - \sum_{i\notin S\in Z} \frac{C_s v_{ik}(S)}{n-s}$$

$$= \sum_{\substack{j\in J\\ n>j>k}} \frac{C_j}{j}\binom{n-1}{j-1} - \sum_{\substack{j\in J\\ n>j>k}} \frac{C_j}{n-j}\binom{n-1}{j} - \frac{C_k}{n-k}\binom{n-1}{k} + \frac{1}{n}$$

$$= -\frac{C_k(n-1)!}{(n-k)!k!} + \frac{1}{n} = 0$$

Thus $C_k = \frac{(n-k)!k!}{n!}$. So, $C_i(S) = \frac{(n-s)!(s-1)!}{n!}$ if $i\in S$ and

$C_i(S) = \frac{(n-s-1)!s!}{n!}$ otherwise. Putting these constants into the formula gives us the equation that we desired and thus this value is unique. Now we need to show that these properties hold on 0--monotonic symmetric PDGs. Additivity, efficiency and symmetry follow by inspection. To show dummy we must go through several stages.

First, we will show that the extreme points of (0,1) normalized monotonic PDGs with player i dummy are exactly the simple monotonic PDGs with player i dummy. The simple monotonic PDGs with player i dummy are obviously extreme points. Then all we need to show is

Use "worth" to refer to a $v(S)$ instead of "value" which is synonymous with "allocation method"!

that all non-simple PDGs with player i dummy are not extreme points. Pick any non-simple PDG v. Pick any value on v not equal to zero. Define a pair of games v+ and v- by adding or subtracting ϵ, respectively from the coalitions with this particular value. Player i is dummy in both v+ and v-, so v is a convex combination of two other games in this class and thus is not an extreme point. True, but not obvious

All we need to show is that if v is a monotonic simple PDG with player i dummy then $\phi_i(v) = 0$. The game v can be of one of two forms: first the set of v_{ij}'s used above, and the unanimity games on all coalitions that have ~~there~~ its supersets contained in Z. The first type obviously have the dummy property, and the second set also has the dummy property since dummy holds on them in fully defined games. This is not completely correct as the proof is not obvious

Putting these pieces together using additivity we know that dummy holds for all symmetric monotonic PDGs. Thus this value is the unique value on this class of games that is efficient, additive, symmetric and have the dummy property. IN FACT, THIS IS WRONG! □

Lemma: Fair ranking, coalitional strategic equivalence and triviality hold on PDGs for ϕ.

Proof: First, we'll show ~~fair~~ triviality. Let v_o be the trivial game. (i.e. $v_o(S) = 0\ \forall S \in Z$.) $\phi_i(v_o) = \Sigma_{S\in Z}\, c_i(S)v_o(S) = 0$.

Next, to show fair ranking. Pick an $T \subseteq N$, and a PDG v. Pick any PDG w with $w(S) = v(S)$ for all $S \in Z$ with $T \neq S$. Assume $\phi_i(v) > \phi_j(v)$ with $i,j \in T$. $\phi_i(v) = \Sigma_{S\in Z}\, c_i(S)v(S) = c_i(T)v(T) + \Sigma_{T\neq S\in Z}\, c_i(S)w(S) = \phi_i(w) + c_i(T)[v(T)-w(T)] = \phi_i(w) + C_{|T|}[v(T)-w(T)]$. Similarly, $\phi_j(v) = \phi_j(w) + C_{|T|}[v(T)-w(T)]$. Since $\phi_i(v) > \phi_j(v)$, $\phi_i(w) + C_{|T|}[v(T)-w(T)] > \phi_j(w) + C_{|T|}[v(T)-w(T)]$. Thus $\phi_i(w) > \phi_j(w)$ and ϕ shows fair ranking.

Finally, to show coalitional strategic equivalence. Pick any 0-monotonic, symmetric PDG v and any $T \subseteq N$ and $\alpha \in \Re$. Define the PDG u, where $u(S) = \alpha$ if $S \supseteq T$ and 0 otherwise. The game u is 0-monotonic so we can define the game $w = u + v$ and calculate its Shapley value. For coalitional strategic equivalence to hold $\phi_i(v) = \phi_i(w)$ for all $i \notin S$. By additivity, $\phi_i(w) = \phi_i(v) + \phi_i(u)$. So, all we need to show is that $\phi_i(u) = 0$ for all $i \notin T$. But player i is a dummy player in the game u so $\phi_i(u) = 0$. WRONG!

$J = \{1,2,4\}$ $T = \{1,2\}$ $w =$ trivial game $\phi_3(w) \neq 0 \neq 1/6 = \phi_3(v)$

This value seems to have all of the same properties as the Shapley value in normal games, and thus appears to be a logical extension to PDGs. The next step is to figure out if Chun's characterization also appl~~y~~ies to this value, we know by the lemma that the value has the necessary properties, but we need still need to show uniqueness. An interesting fact about this value is

This is not stated precisely, and I now think it is incorrect. Let $Z = \{1,2,3,4,12,23,34,41,1234\}$, $v(1234) = v(23) = v(34) = 1$, $v(S) = 0$ otherwise is v_{12} but 1 is not a dummy (we can set $\hat{v}(124) = 1/2$), so, the value is not unique.

that if you can find a monotonic cover for your PDG in which the remaining values that you fill in only depend on the cardinality of the set that you are working with, then the Shapley value on this cover is equal to the value on the PDG. In the future I hope to figure out if this value can be generalized to deal with PDGs that have less restrictions to the structure of Z, the collection of subsets of N. It seems logical that if every player is represent an equal amount in the subsets of equal cardinality, then the formula for this value can easily be generalized to deal with this case. Another possible structur~~ed~~ ~~of~~ for Z that an algorithm might be found to calculate this value, is when the PDG has a subgame for which we have complete information. Both of these ideas are possible directions to explore to extend this value to deal with PDGs with less structure.

This needs justification.

But, we know that we cannot generalize this formula to deal with all games, because we lose uniqueness. Consider the class of 0-monotonic PDGs with Z = {{1},{2},{3},{4},{12},{1234}}. This game has a two parameter family of values that satisfies our properties. Using additivity, efficiency, and symmetry we get $\varphi_1 = \varphi_2 = a\,v(N) + b\,v(\{12\})$ and $\varphi_3 = \varphi_4 = c\,V(N) + d\,v(\{12\})$ where $2a + 2c = 1$ and $b + d = 0$. But, there are no games on this class that have any dummy players, so this formula cannot be generalized any further.

for zero normalized games

nice!

You might wonder why I have restricted the class of games to possess 0-monotonicty instead of monotonicty. If we don't have 0-monotonicty we begin to have problems with existence of our value. Consider that class of monotonic symmetric PDGs on the set $J = \{1,2,n\}$. By additivity, symmetry and efficiency $\phi_1 = 1/n$ $v(N)$ $+ (n-1)a \quad v(1) - a \sum_{i\neq 1} \quad v(i) + \frac{1}{2}(n-1)(n-2)b \sum_{i\neq 1} \quad v(1i) - (n-1)b$ $\sum_{i,j\neq 1} \quad v(ij)$. Using the simple monotonic games that have player 1 dummy we find a and b. But if you consider the game where all players except player 1 are dummy and combine it with the prior results you find that n must be equal to 2. But this is supposed to work for all n, so we have found an example where we have a non-existent value.

what game is this?

Even though at this point we might not be able to generalize the structure of our game, we can still estimate what we think a value should be. In general, we take our PDG and add or delete values until it is symmetric. So if Z has a symmetric structure with the addition of a small number of coalitions then a possible strategy is to consider the symmetric game with these coaltions excluded. Thus, even if we wouldn't be able to find a unique value with a more complex formula, we at least have a reasonable estimate. In the same way we could add coalitions with estimated values until it is symmetric and then find the value for this game.

There are other ways to consider a value on PDGs that is similar to the Shapley value in normal games. Consider the set of all possible superadditive games that agree with a particular superadditive PDG on all of Z. The set of Shapley values for these games give a convex set of values. It would seem sensible to use the geometric center of this convex set as a value for the PDG. In games up to four players this geometric center is exactly the value calculated by the formula I presented earlier. So it seems that this value has even more intuitive sense to it that just the axioms alone. I still don't know if these two values overlap for all games, which I hope to figure out at some future time.

Since there is such a logical extension of the Shapley value to PDGs, one can assume that a similar extension should extend the nucleolus and other values. With any luck, one of these new values might have be easier to calculate for PDGs that have very little natural structure. For example, in finding the nucleolus we might not find a unique value but we could very easily calculate a region of values that have the same properties as the nucleolus has under normal situations.

This area still has many directions in which we can head. One of this deals with what classes of PDGs has a value with these properties, and whether it is unique. Another deals with finding addtional axioms which will quarentee uniqueness and modifications on current axioms to get uniqueness. In the future, similar techniques to what I did with the Shapley value can be used to find extensions for other standard values.

proof?

References

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Dear Dave,

I hope that you have been able to relax after eight weeks of mathematics (with two jobs, I don't know how you could). I have had administrative work for the Council on Undergraduate Research, lectures for the New Jersey Governor's School, and software library development for Introductory Statistics. Jeanne and I took a few hours off for our anniversary, but now I'm back finishing up REU stuff.

I have enclosed (1) a request for an evaluation of the program, (2) the original of your REU report upon which I have written a number of suggestions, and (3) a copy of participants' whereabouts. Please return your evaluation to me by September 17. If you would like a revised copy of your report to be sent to the NSF and other interested persons, please return your revision by September 17. I have held off sending out copies of student reports pending each student's decision whether or not to revise. There is no requirement to revise your report; it is up to you based upon your time and interest. In late September, I will send you copies of the other reports. If there are any articles I discuss below which you do not have, I can also send you copies. Just let me know.

You have broken ground into an important, new area of research. You have generated a number of original and highly creative ideas; however, the details are tricky. With the benefit of more time and a relaxed atmosphere, I have discovered two serious errors in your proof of Theorem 1. Everything centers around the dummy axiom and zero monotonicity. The key error in the uniqueness proof is that v_{i1} is *not* zero monotonic because the worth of the grand coalition in the zero normalized form is $2 - n$ which is less than 0. This error can be corrected as the following lemmas show.

Lemma 1. Player i is a dummy in the partially defined game v_{ik} with respect to monotonic extensions if $1 \leq k < n$, with respect to zero-monotonic extensions if $1 < k < n$, and with respect

$v_{ik}(S) = 1$ if $|S| > k$ or
$|S| = k$ and $i \notin S$
0 otherwise

$k \in J$

to superadditive extensions if $n/2 < k < n$.

Proof. Suppose $\hat{v}$ is a monotonic extension of v_{ik}. Since the worth of singletons is at least 0 and $v(N) = 1$, it follows that $0 \leq \hat{v}(S) \leq 1$ for all coalitions S. If $|S| < k$, then there exists $T \in Z$ satisfying $S \subseteq T$, $i \in T$, and $|T| = k$; this implies that $\hat{v}(S) \leq v(T) = 0$, and so $\hat{v}(S) = 0$. If $|S| > k$, then there exists $R \in Z$ satisfying $R \subseteq S$, $i \notin R$, and $|R| = k$; this implies that $\hat{v}(S) \geq v(R) = 1$, and so $\hat{v}(S) = 1$. Hence, there is only one monotonic extension of v_{ik}, and i is clearly a dummy in $\hat{v}$. Because v_{ik} is zero normalized when $1 < k < n$, zero-monotonic and monotonic extensions are the same. Because the set of superadditive extensions for a partially defined game is always a subset of the zero-monotonic extensions, the superadditive and zero-monotonic extensions for v_{ik} are the same when $n/2 < k < n$. Note that v_{i1} is not zero-monotonic, and v_{ik} is not superadditive for $k \leq n/2$.

Lemma 2. Suppose i is a player. Player $j \neq i$ is a dummy in the unanimity on $\{i\}$ partially defined game with respect to zero-monotonic or superadditive extensions.

Proof. The zero normalization of the unanimity on $\{i\}$ partially defined game has all zero worths. So, the unique zero-monotonic or superadditive extension is the unanimity on $\{i\}$ game, and clearly $j \neq i$ is a dummy in this extension.

Lemma 3. Suppose T is a nonsingleton coalition. Player $j \notin T$ is a dummy in the unanimity on T partially defined game with respect to superadditive extensions if and only if all supersets of T are contained in Z.

Proof. Suppose v is the unanimity on T partially defined game, $\hat{v}$ is a superadditive extension. Since the worth of singletons is ~~at least~~ 0 and $v(N) = 1$, it follows that $0 \leq \hat{v}(R) \leq 1$ for all coalitions R. If $|R| < |T|$, then there exists $S \in Z$ satisfying $R \subseteq S \neq T$ and $|S| = |T|$; this implies that $\hat{v}(R) \leq v(S) = 0$, and so $\hat{v}(R) = 0$. So, if all supersets of T are contained in Z, the unique superadditive extension is the unanimity on T game, and clearly $j \notin T$ is a dummy in this extension. If all supersets of T are not contained in Z, then there exists $R \notin Z$ satisfying $|R| > |T|$, $j \in R$, and both $T \backslash R$ and $R \cap T$ are nonempty. Let $\hat{v}$ be the unanimity on T game except $\hat{v}(R) = 1/2$. Clearly, $\hat{v}$ is a superadditive extension of v, and j is not a dummy.

and "supersets"

Lemmas 1-3 can be used to show uniqueness (assuming we already have existence) of a value on partially defined games satisfying efficiency, symmetry, linearity, and dummy. With respect to monotonic extensions, the formula given in your report is obtained. With respect to zero-monotonic extensions, the following formula is obtained:

$$\phi_i(v) = \sum_{\substack{S \in Z(i) \\ |S|>1}} \frac{(n-s)!\,(s-1)!}{n!}\,[v(S) - \sum_{j \in S} v(j)] \;-\; \sum_{\substack{S \in Z \backslash Z(i) \\ |S|>1}} \frac{(n-s-1)!\,s!}{n!}\,[v(S) - \sum_{j \in S} v(j)]$$

where $Z(i) = \{ S \in Z : i \in S \}$. With respect to superadditive extensions, Lemmas 2 and 3 could be used to find a unique formula if and only if Z is of the form $\{ S \subseteq N : |S| = 1, k, k+1, \ldots, n\}$. At least this is my conjecture at this point.

As you note in your report, these formulas clearly satisfy efficiency, symmetry, and additivity. The difficulty is showing that the dummy property holds. The key error is in the characterization of all simple partially defined games in which player i is a dummy. For awhile, I thought that I had a patch, but then I found the following example.

Example. Let $J = \{1, 2, 4, 5\}$ and v is the simple partially defined game with the following winning coalitions: 12345, 1234, 1235, 1245, 1345, 12, 13, 14. There is a unique (zero-) monotonic extension, because the coalitions 123, 124, 125, 134, 135, 145 must be winning (supersets of 12, 13, or 14), and the coalitions 234, 235, 245, 345 must be losing (subsets of 2345). Player 5 is a dummy; however, the formula yields $\phi_5(v) = 1/20 \neq 0$.

Thus, there is no value on all partially defined games with monotonic or zero-monotonic extensions. I conjecture that existence can be proved for the monotonic and zero-monotonic cases when J is of the form $\{1, k, k+1, \ldots, n\}$. The argument would go as follows. For zero-normalized games when J is of the given form, the formula for player i can be written as a weighted sum of marginals for i plus a weighted sum of $v(S)$ where each S contains i and is of size k. Now if i is a dummy, all of the terms described above are zero. In order to complete the proof for the zero monotonic case, note that i is a dummy in a partially defined game v with respect to zero-monotonic extensions if and only if i is a dummy in the zero normalization of v with respect to zero-monotonic extensions. This last statement is not true for monotonic extensions; however, I believe that there may be another approach for this case. If my conjecture is correct, then there is something wrong with your example on the top of the next to last page of your report. Actually, it was my trying to figure out what "the game where all players except player 1 are dummy" meant that motivated the preceeding work. This example gives some hope that a unique value may exist for superadditive extensions, at least for Z with particular structures.

I think that your work, if continued, could result in a very nice, publishable paper. The monotonic, zero-monotonic, and superadditive cases need to be examined throughly using the axiomatics. This might be sufficient, but it would be better to obtain general connections with the geometric approach. Let me know what you plan to do. I would be interested in collaborating with you if you wish.

I will close with a few words about recommendations. I would be happy to write you a recommendation for another summer program, graduate study, or employment upon request. It is my policy to always share with you a copy of my letter of recommendation for you. If there is sufficient time between your request and the receipt deadline, I will send you a first draft for comment. You received two strong letters of recommendation when you applied to the REU program.

Good luck digesting all of this!

Sincerely,

David Housman