Final Report

Game Theory REU: May 29-July 20, 1990

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PART I:

AN ALLOCATION METHOD FROM INDIVIDUAL COMPLAINT FUNCTIONS

0. Purpose

This paper defines an individual complaint function, which is a variant on nucleolus excess functions. By setting these to be equal (which is equivalent, as we shall see, to minimization), we find a class of allocation methods μ. This class of values is characterized by efficiency, symmetry, additivity, as well as an additional property, the inessential game property, which states that allocations will allocate $v(i)$ to player i on an inessential game. A few known values are in this class, including the Shapley value.

1. Definitions

A *cooperative game* we define to be a pair (N,v). N is any finite set, usually $\{1,2,\ldots,n\}$,[1] whose elements are called *players*; v is a real valued function on all subsets $S \subseteq N$, which are referred to as *coalitions*, and v must satisfy $v(\emptyset)=0$. An

[1]Throughout, we will use n and s to indicate the cardinality of sets N and S respectively.

allocation is a vector $x=(x_1, x_2, \ldots, x_n)\in\mathbb{R}^n$. Intuitively, $v(S)$ for $S\subseteq N$ is the worth of coalition S, that is, the return the players in S can expect by cooperating with each other; and an allocation assigns a return to individual players. An *allocation method* ψ is a function which assigns an allocation vector $\psi(v) = (\psi_1(v),\ldots,\psi_n(v))$[2] to a given game v.

An allocation is *individually rational* if $x_i\geq v(i)$[3] $\forall i\in N$, that is, no player receives less than he might receive alone. An allocation is *efficient* if $v(N) = \Sigma x_i$, $i\in N$, that is, if the entire value $v(N)$ is distributed among the players. An *imputation* is an allocation satisfying efficiency and individual rationality.

An allocation method ψ is *efficient* if $\psi(v)$ is efficient for all games v. ψ is *additive* if $\psi(v+w) = \psi(v)+\psi(w)$, for all games v and w which are defined on the same set of players, where $v+w$ is the game defined by $(v+w)(S) = v(S)+w(S)$. ψ is *linear* if $\forall\lambda\in\mathbb{R}$, $\psi(v+\lambda w) = \psi(v)+\lambda\psi(w)$, where λw is the game defined by $(\lambda w)(S)=\lambda w(S)$. Note that linearity implies additivity. ψ is *symmetric* if $\forall\sigma\in\Pi_n$ (the nth permutation group) and for all games v, we have $\psi_i(\sigma v) = \psi_{\sigma(i)}(v)$ $\forall i\in N$, where σv is defined by $\sigma v(S)=v(\sigma(S))$. We say player $i\in N$ is a *dummy player* if $\forall S\ni i$, $v(S)=v(S-i)$. An allocation method ψ satisfies the *dummy property* if for all games v, $\psi_i(v)=0$ for all dummy players i in game v.

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[2] ψ and $(\psi_1,\ldots,\psi_n)$ will be used when the game is understood.

[3] Set brackets will be omitted for singleton sets when the meaning is clear.

One commonly examined allocation method is the nucleolus. We define the *excess* $e(x,S)$ of a coalition $S\neq\emptyset$ relative to an allocation x to be $v(S)-x(S)$;[4] the *excess vector* $e(x)$ relative to an allocation x is the 2^n-1 vector of all coalitional excesses, whose entries are listed in descending order. It is a well known result that there is a unique ~~allocation~~ imputation which minimizes the excess vector lexicographically, and the *nucleolus* is defined to be this value.

2. Construction of the Class of Values μ

In this paper, we attempt to characterize a class of values which we define as a variant on the idea behind the nucleolus. Rather than examining excesses of coalitions relative to allocations, we construct complaints of individuals relative to coalitions. Define a *complaint function* to be a real valued function which operates on a given allocation and individual, which is of the form

$$\mathrm{comp}(x,i)=\sum_{\substack{S\subseteq N\\ i\in S}} c_S[v(S)-x(S)]+\sum_{\substack{S\subseteq N\\ i\notin S}} d_S[v(S)-x(S)]$$

where $c_k, d_k \in \mathbb{R}\ \forall k\in\{1,2,\ldots n\}$. Intuitively, we are summing excesses weighted by fixed weights c_k and d_k which depend on the size of S and the presence or absence of i in S. This is the amount player i dislikes allocation x, relative to the particular

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[4]Throughout, we will use $x(S)$ to indicate Σx_i, $i\in S$.

weights.

We use $x(S)=\Sigma x_i$ $i\in S$, let $k=|S|$, and collect terms to get that comp(x, i)=

$$\sum_{i\in S} c_s v(S) + \sum_{i\notin S} d_s v(S) - x_i \sum_{k=1}^{n} \binom{n-1}{k-1} c_k - \left[\sum_{j\neq i} x_j\right]\left[\sum_{k=2}^{n} \binom{n-2}{k-2} c_k + \sum_{k=1}^{n-1} \binom{n-2}{k-1} d_k\right]$$

$$= \sum_{i\in S} c_s v(S) + \sum_{i\notin S} d_s v(S) - x_i \sum_{k=1}^{n} \binom{n-1}{k-1} c_k - \left[\sum_{j\neq i} x_j\right]\left[\sum_{k=1}^{n} \binom{n-1}{k-1} \frac{(k-1)c_k+(n-k)d_k}{n-1}\right]$$

With an efficient allocation x, we then get Σcomp(x, i), $i\in N$ =

$$\sum_{S\subseteq N} [sc_s+(n-s)d_s]v(S) - v(N)\sum_{k=1}^{n} \binom{n-1}{k-1}(kc_k+(n-k)d_k)$$

As with the nucleolus, we would like to define an efficient allocation method by minimizing the maximum of the n complaints. However, this is particularly easy because, as we can see, the n complaints sum to a value dependent only on the game, and not on x, for x efficient. Therefore, we can define a class of allocation methods dependent on the choice of weights. μ will be a function whose value is an efficient allocation method and whose arguments are vectors c and d, defined by comp(μ,i) = [Σcomp(x,j)]/n where x is any efficient allocation, the sum ranges over all $j\in N$, and c and d are the vectors of c_i and d_i respectively. Substituting and simplifying, and also using $[v(N)-x_i] = \Sigma x_j$, $j\neq i$, we get

$$\mu(c,d)_i = \frac{1}{\gamma}\left[\sum_{S \ni i}(c_s - d_s)\,v(S) - \sum_{S \subseteq N}\frac{s}{n}(c_s - d_s)\,v(S)\right] + \frac{v(N)}{n}$$

$$\text{where } \gamma = \sum_{k=1}^{n-1}\binom{n-2}{k-1}(c_k - d_k)$$

There are two important points to note about this solution for $\mu(c,d)$. First, γ is merely a normalizing factor. (We shall see later that it normalizes $\mu(c,d)$ to get an allocation method which satisfies what we will later define as the inessential game property.) Indeed, for a given $c_1 \geq d_1$, $c_2 \geq d_2$, ..., $c_n \geq d_n$, define $c'_k = c_k/\gamma$ and $d'_k = d_k/\gamma$, $\forall k \in \{1,\dots,n\}$. It is easily seen that $\gamma' = 1$ and that $\mu(c',d')_i = \mu(c,d)_i$, $\forall i \in N$.

The other important point to note about the formula for $\mu(c,d)$ is that it is only dependent on the n-1 numbers $c_1 - d_1$, $c_2 - d_2$, ..., $c_{n-1} - d_{n-1}$. Notice the formula depends only on these differences, and $c_n - d_n$ can be eliminated from the formula by requiring the summations to be for $S \neq N$, which won't change the values. We will thus consider c and d to henceforth be n-1 vectors. Further, we define $\mu(c) \equiv \mu(c,0)$ where 0 is the zero vector of proper size. Since $\mu(c-d) = \mu(c,d)$, we will not lose generality by restricting our inquiries to $\mu(c)$. Henceforth, then, we will use

$$\mu(c)_i = \frac{1}{\gamma}\left[\sum_{S \ni i} c_s\,v(S) - \sum_{S \subseteq N}\frac{s}{n}\, c_s\,v(S)\right] + \frac{v(N)}{n} \qquad \text{(eq.1)}$$

where $\gamma = \sum_{k=1}^{n-1} \binom{n-2}{k-1} c_k$ (eq.2)

$$\mathrm{comp}(x,i) = \sum_{S \ni i} c_s [v(S) - x(S)] \qquad \text{(eq.3)}$$

3. An Axiomatization of $\mu(c)$

Let $c \in \mathbb{R}^{n-1}$. It should be clear that $v(N) = \Sigma\mu(c)_i$, $i \in N$; also, certainly $\mu(c)_i(\sigma v) = \mu(c)_{\sigma(i)}(v)$ $\forall \sigma \in \Pi_n$. Furthermore, since $\forall i$, $\mu(c)_i$ is a linear combination of $\{v(S): S \subseteq N\}$, we see $\mu(c)(v+\lambda w) = \mu(c)(v) + \lambda\mu(c)(w)$ for any games v and w over the same set of players and $\forall \lambda \in \mathbb{R}$. Therefore, we see that for any c, $\mu(c)$ is efficient, symmetric, and linear.

$\mu(c)$ has one additional property, as follows. A game v is an *inessential game* if $\forall S \subseteq N$, $v(S) = \Sigma v(i)$, where the sum ranges over all $i \in S$.[5] We define an allocation method ϕ to have the *inessential game property* if $\phi_i(v) = v(i)$ for all inessential games v. It is clear $\forall c \in \mathbb{R}^{n-1}$, $\mu(c)$ satisfies the inessential game property because if $x = (v(1), \ldots, v(n))$, the n complaints $\mathrm{comp}(x,i)$ will all be zero on an inessential game. Because $\mu(c)$ is the unique allocation making the complaints equal, $\mu(c) = x$. Thus $\mu(c)$ satisfies the inessential game property.

In fact, linearity, symmetry, efficiency, and the inessential game property uniquely characterize the class of values $\mu(c)$. To show that these axioms determine an allocation

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[5]Inessential games are also known as *additive games*.

values $\mu(c)$. To show that these axioms determine an allocation in the form $\mu(c)$, let ψ be an allocation satisfying linearity, symmetry, efficiency, and the inessential game property. By linearity, $\forall i\in N$, ψ_i is a linear combination of $\{v(S): S\subseteq N, S\neq\{\emptyset\}\}$. To see this, let $\{w_S: S\subseteq N, S\neq\emptyset\}$ be the standard basis for games, that is, $w_S(S)=1$ and $w_S(T)=0$ for $T\neq S$. Then for any game v, $v=\Sigma v(S)w_S$, where the sum is over all nonempty $S\subseteq N$. By linearity, $\psi_i(v)=\Sigma\psi_i(w_S)v(S)$. Thus indeed ψ_i is a linear combination of $\{v(S)\}$, with the coefficient of each $v(S)$ given by $\psi_i(w_S)$.

Furthermore, let S and T be any coalitions of the same cardinality which both contain some fixed $i\in N$. We can find a permutation σ on N which leaves i fixed and for which $\sigma(S)=T$. Then $\sigma(w_S)=w_T$. By symmetry, and since i is fixed under σ, $\psi_i(w_T)=\psi_i(\sigma w_S)=\psi_{\sigma(i)}(w_S)=\psi_i(w_S)$. Similarly, $\sigma(w_{S-i})=w_{T-i}$, and so $\psi_i(w_{T-i})=\psi_i(w_{S-i})$. Thus, the coefficient of any $v(S)$ in ψ_i depends only on the size of S and whether $i\in S$. Clearly, then, we can find real numbers $a_1,\ldots,a_n$ and $b_1,\ldots,b_{n-1}$ such that

$$\psi_i = \sum_{S\ni i} a_s v(S) + \sum_{i\notin S} b_s v(S)$$

By efficiency, we know $a_n=1/n$ and for $k<n$, $ka_k+(n-k)b_k=0$. Let $c_k=nb_k/k$. Then $b_k=kc_k/n$ and $a_k=-(n-k)b_k/k=-(n-k)c_k/n$. Thus,

$$\psi_i = \sum_{S\ni i} \frac{n-s}{n} c_s v(S) - \sum_{i\notin S} \frac{s}{n} c_s v(S) + \frac{v(N)}{n}$$

$$= \sum_{S \ni i} c_s v(S) - \sum_{S \subseteq N} \frac{s}{n} c_s v(S) + \frac{v(N)}{n}$$

Comparing this to (eq. 1), it is clear that we will have $\psi_i = \mu(c)$ once we show $\gamma=1$, for γ as defined by (eq. 2).

To show this is the case, fix any $i,j \in N$, $i \neq j$. Let v be the inessential game with $v(j)=1$ and $v(j')=0$ $\forall j' \neq j$. By the inessential game property, $\psi_i = v(i) = 0$. Thus,

$$0 = \sum_{S \supseteq \{i,j\}} c_s - \sum_{S \ni j} \frac{s}{n} c_s + \frac{1}{n} = \sum_{k=1}^{n-1} \left[\binom{n-2}{k-2} - \frac{k}{n} \binom{n-1}{k-1} \right] c_k + \frac{1}{n}$$

Multiplying through by n and rearranging terms, we get

$$1 = \sum_{k=1}^{n-1} \left[\frac{(n-1)!\,k}{(n-k)!\,(k-1)!} - \frac{(n-2)!\,n}{(n-k)!\,(k-2)!} \right] c_k = \sum_{k=1}^{n-1} \binom{n-2}{k-1} c_k$$

Thus we see that the inessential game property indeed forces $\gamma=1$. Therefore, we see that our axioms characterize the class of values $\mu(c)$ uniquely. We have now established the following.

<u>Theorem</u>
An allocation sets equal the complaint functions (eq. 3) for some c if and only if it satisfies efficiency, symmetry, linearity, and the inessential game property.

For any c, we now know that the class of values $\mu(c)$ is exactly those values which satisfy efficiency, symmetry, linearity, and the inessential game property. These properties, moreover, are not redundant. To see this, consider the following allocation methods:

$\alpha(v)_i = v(i)$

Fix a $j \in N$
$\beta(v)_i = v(i)$ for $i \neq j$
$\beta(v)_j = v(N) - \Sigma v(i)$, summing $\forall i \neq j$

$\eta(v)_i = v(i)$ for v an inessential game
$\eta(v)_i = 1/n$ for v not an inessential game

$\theta(v)_i = 1/n$

Each of these allocation methods fails on exactly one of our four properties. α is not efficient, β is not symmetric, η is not linear, and θ does not satisfy the inessential game property. So we see our characterization of $\mu(c)$ having these four properties is a non-redundant characterization.

4. Some Specific Allocation Methods $\mu(c)$

It still remains to be seen what vectors c (equivalently, what vectors c and d) will yield useful allocations. One guess, for instance, would be to assign $c_k = 1/k$ $\forall k \in \{1,...,n-1\}$, reasoning as follows: if coalition S is imposed, each player in that coalition can expect to receive 1/s more than under the current allocation, so player i's objection is the sum of these objections over all coalitions.[6] This choice of c leads to

$$\mu(c)_i = \frac{n-1}{2^{n-1}-1}\left[\sum_{S \ni i} \frac{v(S)}{s} - \sum_{S \subseteq N} \frac{v(S)}{n}\right] + \frac{v(N)}{n}$$

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[6]Dividing by s is similar to the definition of the *per capita nucleolus* which is defined analogously to the nucleolus, except using *per capita excesses* $ex'(x,S) = ex(x,S)/s$.

A few well known allocation methods can be characterized as $\mu(c)$ for some c. For instance, $c_1=1$ and $c_k=0$ for $k\neq 1$ leads to

$$\mu(c)_i = v(i) + \frac{1}{n}\left[v(N) - \sum_{j\in N} v(j)\right]$$

which is known as the *equal allocation of joint venture value.* Conversely, setting $c_{n-1}=1$ and $c_k=0$ for $k\neq n-1$ leads to

$$\mu(c)_i = s_i + \frac{1}{n}\left[v(N) - \sum_{j\in N} s_j\right]$$

which is known as the *equal allocation of nonseparable value,* where $s_j=v(N)-v(n-j)$ is the *separable value* of player i. Finally, by defining

$$c_i = \frac{1}{\binom{n-2}{i-1}}$$

We get the *Shapley Value* φ, namely

$$\varphi_i = \sum_{S\ni i} \frac{(s-1)!(n-s)!}{n!}[v(S) - v(S-i)]$$

It is easily seen that the Shapley Value for player i is the average its marginal contribution v(S)-v(S-i) over all orderings of the players in N, and so this is a generalization of the equal allocation of joint value and the equal allocation of nonseparable value, the former limiting itself to summing over coalitions of size one, the latter over size n-1.

It is a well known result, proved by Lloyd Shapley, that the Shapley value is the unique allocation method which satisfies efficiency, additivity, symmetry, and the dummy property. By our theorem, then, we have

Corollary
The Shapley value is the unique value which is an optimal solution to the equations (eq. 3) and which satisfies the dummy property.

5. Further Research Areas

In the corollary, we should be able to reach the same conclusion with a property slightly less strong than dummy, such as a condition which when combined with the inessential game property will yield the dummy property. It is not difficult to mathematically define what we want, but we currently lack an intuitive axiom which will suffice.

On a completely different track, consider replacing the summation in our definition of complaint, (eq. 3), to a maximum, and letting c=(1,1,...,1). We will achieve a non-uniquely determined imputation anywhere in the *least core* which is the set of allocations which minimize the maximum excess (not lexicographically). By altering this new complaint function it may be possible to recover the nucleolus.