Final Report

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Michael Maltenfort

PART II:

CONSISTENCY ON PARTITION FORM GAMES

# 1. Definitions

## 1.1 Partitions

A *partition* P of a set T is a set of nonempty subsets of T which are pairwise disjoint and whose union is T.[1] Suppose P and Q are partitions of sets A and B respectively. We say P and Q are *disjoint* if $A \cap B = \emptyset$. Throughout the remainder of this section, P and Q will be disjoint partitions of sets A and B respectively.

We define a *joint partition R from P and Q* to be a set R of ordered pairs (p,q) where $p \in P \cup \{\emptyset\}$, $q \in Q \cup \{\emptyset\}$, and $(\emptyset,\emptyset) \notin R$, such that each element p of partition P appears exactly once in a pair (p,q), and each element q of partition Q appears exactly once in a pair (p,q). We define the *partition from R* to be $\text{part}(R)=\{p \cup q : (p,q) \in R\}$. We see that part(R) is a partition of $A \cup B$. For example, if P={{a,b},{c}} and Q={{X,Y}}, then the three

[1] In the degenerate case, $\emptyset$ is the only partition of $\emptyset$.

joint partitions from P and Q are $\{(\{a,b\},\{X,Y\}),(\{c\},\emptyset)\}$, $\{(\{a,b\},\emptyset), (\{c\},\{X,Y\})\}$, and $\{(\{a,b\},\emptyset),(\{c\},\emptyset),(\emptyset,\{X,Y\})\}$. The partitions from these joint partitions are $\{\{a,b,X,Y\},\{c\}\}$, $\{\{a,b\},\{c,X,Y\}\}$, and $\{\{a,b\},\{c\},\{X,Y\}\}$ respectively.

We define a *numeric partition* of an $n \in \mathbb{R}^+$ to be an unordered sequence[2] of positive integers $a_1,\ldots,a_k$ with $\Sigma a_i = n$.[3] If X and Y are numeric partitions of n and m respectively, we define a *joint numeric partition from X and Y* to be an unordered sequence of ordered pairs of nonnegative integers $(a_1,b_1),\ldots,(a_k,b_k)$ where the nonzero first coordinates give the numeric partition X, the nonzero second coordinates give the numeric partition Y, and there are no pairs (0,0).

Partitions and joint partitions induce numeric partitions and joint numeric partitions respectively, as follows. If $P=\{p_1,\ldots,p_i\}$, define the *numeric partition corresponding to P*, written num(P), to be the unordered sequence $\#p_1,\ldots,\#p_i$; notice this is a numeric partition of #A. In addition, suppose $R=\{(p_1,q_1),\ldots,(p_j,q_j)\}$ is a joint partition from P and Q. Then define the *joint numeric partition corresponding to R* to be numj(R)= $(\#p_1,\#q_1),\ldots,(\#p_j,\#q_j)$. We see that numj(R) is a joint numeric partition from num(P) and num(Q).

If X and Y are numeric partitions, we define a *mixture of P and Q* to be a function m on the set of joint numeric partitions

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[2]i.e. a sequence where order is unimportant

[3]In the degenerate case where n=0, the only numeric partition is "the empty sequence," a sequence with no elements.

from num(P) and num(Q) into $\mathbb{R}^+$ such that $\Sigma m(\overline{numj}(R))=1$, where the sum ranges over all joint partitions R. We think of m as giving weights to the joint partitions.

Finally, we define a *mixture method* M to be a function which assigns a mixture M(P,Q) to every pair of disjoint partitions P and Q, with the following symmetry property. If P′ and Q′ are disjoint partitions such that num(P)=num(P′) and num(Q)=num(Q′), then M(P,Q)=M(P′,Q′).

Although these definitions seem lengthy and tedious, they are necessary. In order to sum over all joint partitions R from P and Q (rather than all numeric joint partitions from num(P) and num(Q)), we must define mixtures and mixture methods on P and Q, rather than on num(P) and num(Q) as might seem more natural. This symmetry condition insures that mixture methods will actually depend on num(P) and num(Q).

## 1.2 Partition Form Games

A *partition form cooperative game*, which we will refer to simply as a game, is a pair (N,v). N is any finite set, usually {1,2,...,n} whose elements are called *players*; nonempty subsets of N are called *coalitions*.[4] V is a function which assigns a real number to every coalition S and partition of N−S. Intuitively, the real number v(S;P) is the worth of coalition S

[4]In other literature, ∅ is considered a coalition, with v(∅)=0. Here, however, we choose to define coalitions to be nonempty.

cooperating when the remaining players N−S cooperate in the coalitions specified by partition P.[5] Because notation becomes difficult, we adopt the following conventions for a game with a small number of players: coalitions will be written without spaces, braces, or commas, and partitions will be written without braces, using commas to separate elements of the partition. For example, $v(12;34,5,67)$ means $v(\{1,2\};\{\{3,4\},\{5\},\{6,7\}\})$. Furthermore, if S=N, we will omit writing the empty partition, writing $v(N)$ rather than $v(N,\emptyset)$.

Let $(N,v)$ be a partition form game. Given a coalition T and a partition Q of N−T, our objective is to define a restriction game of v. Given such a T and Q and a mixture method M, we define a *restriction game on T;Q using M* to be a game $(T,v_{T;Q;M})$ where $v_{T;Q;M}(S;P)=\Sigma m(R)v(S;\text{part}(R))$, where $m=M(P,Q)$ and the sum ranges over all joint partitions R from P and Q. Usually, we will assume M to be fixed, and write $v_{T;Q}$ for $v_{T;Q;M}$.

An *allocation* for a game $(N,v)$ is a vector $x=(x_1,x_2,\ldots,x_n)\in\mathbb{R}^n$. An *allocation method* is a function $\mu$ from games to allocations on those games, which we write $\mu(v)=(\mu_1(v),\ldots,\mu_n(v))$, omitting v when it is understood.

Fix a mixture method M. If T is a coalition, and $\mu$ is an allocation method, define the *reduced game on T relative to $\mu$* to be a game $(T,v^{T,\mu})$ where $v^{T,\mu}(S,P)=\Sigma\mu_i(v_{S\cup(N-T);P;M})$, where the sum ranges over all $i\in S$.

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[5]Other literature specifies P to be a partition of N, one of whose elements must be S. When we write $v(S,P)$, this means $v(S,P\cup\{S\})$ in traditional notation.

We say an allocation method is *consistent* if $\mu_i(\overline{v})=\mu_i(v^{T,\mu})$ $\forall T\subseteq N$ and $\forall i\in T$.

## 2. Consistent Allocation Methods for a Three Player Game

Our objective is to find a class of consistent allocation methods for three player games. As a precondition, we stipulate that if (N,v) is a two player game, then $\mu_1=\frac{1}{2}[v(1;2)+v(12)-v(2;1)]$ and $\mu_2=\frac{1}{2}[v(2;1)+v(12)-v(1;2)]$. This is a reasonable condition because a 2-player partition game is equivalent to a 2-player TU game, and this 2-player allocation is standard for all well known values on TU games.

We must choose a mixture method M in order to have our reduced games well defined. First notice that if P or Q is the trivial partition, (the partition $\emptyset$ of the set $\emptyset$), then there is no choice to be made. Suppose Q is the trivial partition. Then $\{(p,\emptyset)|p\in P\}$ is the only joint partition of P and Q. Thus, M(P,Q) is defined only on this one joint partition. But then the value at this joint partition must be 1 by our stipulation $\Sigma(M(P,Q))(R)=1$, because the sum is over only one joint partition. The case where P is trivial is similar.

For games with under four players, there is only one non-trivial case. We see this as follows. Now, we are using M in our restriction game, where $v_{T,Q,M}(S;P)=\Sigma m(R)v(S;part(R))$, $m=M(P,Q)$. Since $S\neq\emptyset$ (being a coalition), $\#S>0$. Q is a partition of N−T, and P is a partition of T−S. Since #N=3 and #S≥1, the only nontrivial partitions P and Q are partitions of singleton sets N−T and T−S. Thus, we need only define M for P and Q partitions of singleton sets. Let P={{i}} and Q={{j}}. (A singleton set can only partition in one way.) There are exactly two joint partitions: {{i,j}} and {{i},{j}}. In order to maintain generality, we define M({{i}},{{j}}) to give (1−a) on {{i},{j}} and a on {{i,j}}, where $0 \le a \le 1$. This clearly gives the most general M.

Then if {i,j,k}={1,2,3}, we will have:

$$
\begin{aligned}
v_{ij;k}(i;j) &= av(i;jk) + (1-a)v(i;j,k) \\
v_{ij;k}(j;i) &= av(j;ik) + (1-a)v(i;j,k) \\
v_{ij;k}(ij) &= v(ij;k)
\end{aligned}
$$

Using our already defined 2-player allocation on the restriction games, we get

$$
\begin{aligned}
v^{12,\mu}(1;2) &= \mu_1(v_{13;2}) = \tfrac{1}{2}[av(1;23)+(1-a)v(1;2,3)+v(13;2) \\
&\qquad -av(3;12)-(1-a)v(3;1,2)] \\
v^{12,\mu}(2;1) &= \mu_2(v_{23;1}) = \tfrac{1}{2}[av(2;13)+(1-a)v(2;1,3)+v(23;1) \\
&\qquad -av(3;12)-(1-a)v(3;1,2)] \\
v^{12,\mu}(12) &= \mu_1(v_{123})+\mu_2(v_{123}) = \mu_1(v)+\mu_2(v)
\end{aligned}
$$

Applying our 2-player allocation to our reduction games, we get

$$\mu_1(v^{12,\mu}) = \tfrac{1}{2}[v^{12,\mu}(1;2)+v^{12,\mu}(12)-v^{12,\mu}(2;1)]$$

By consistency, $\mu_1(v)=\mu_1(v^{12,\mu})$. Substituting for all terms and simplifying gives

$$
\begin{aligned}
\mu_1(v)-\mu_2(v) &= \tfrac{1}{2}[av(1;23)-av(2;13)+(1-a)v(1;2,3)+(1-a)v(2;1,3) \\
&\qquad +v(13;2)-v(23;1)]
\end{aligned}
$$

Using symmetry, we can write a similar expression for $\mu_1(v)-\mu_3(v)$. We then add these two equations, and add v(N) to both sides. Using efficiency, and dividing by 3, we get

$$\mu_1(v) = \tfrac{1}{3}[v(123)-v(23;1)+av(1;23)+(1-a)v(1;2,3)]$$

$$+(1/6)[v(13;2)-(1-a)v(3;1,2)-av(3;12)$$
$$+v(12;3)-(1-a)v(2;1,3)-av(2;13)]$$

Notice that if we set $z=6a$, then this is exactly the class of values which Patrice McCaulley found in her research this summer. She approached this value by requiring it to satisfy partition form extensions of the TU axioms efficiency, symmetry, additivity, and dummy; here we have shown that the same class of values arises from efficiency and consistency. Her variable depended on the choice of an allocation for a specific game, and here our value depends on our mixture method M.

## 3. Further Research Areas

The allocation method presented here should be extended to games with more than three players. In addition, we should try to prove directly that for efficient allocation methods, consistency is equivalent to symmetry, additivity, and dummy (or perhaps "dummy independence," an extension of dummy). The forward direction should not be difficult; proof of dummy and dummy independence relies on showing that dummy players are preserved in restriction and reduced games.