MONOTONICITY OF VALUES FOR COOPERATIVE GAMES

Christine Renee Martell

Holly M. Winn

July 1989

It is important to recognize the power behind this paper. Without a grant by the National Science Foundation this program would not have been available. Equally important, without our organizer, coordinator and advisor David Housman, our insight would not have been as insightful. We would like to take this opportunity to thank David and give him our appreciation for his dedication and motivation.

CRM

HMW

July 1989

*** SECTION ONE ***

THE FOUNDATION

I. INTRODUCTION:

The purpose of this paper is to examine various allocation methods and their respective properties. Specifically, Section Two looks at different values in terms of the property total population monotone (TPM). Values which are TPM on maximal classes of games are characterized. The hope here is to determine optimal methods if TPM is to be considered an important factor in fairness. Section Three begins by looking at aggregate monotonicity on the nucleolus. It further explores the relationship between aggregate monotonicity and group monotonicity of the nucleolus on a special class of games. This section provides definitions of the monotonicity properties and values of interest along with a short review of the literature.

II. DEFINITIONS

An n-person cooperative game is a pair (N, v) where $N = \{1, 2, 3, \ldots n\}$ is the set of players and v is a real-valued function on all nonempty coalitions $S \subset N$. We use the convention that $v(\emptyset) = 0$. We are mainly concerned with superadditive games, that is, games for which $v(S \cup T) \geq v(S) + v(T)$ for all groups S and T satisfying $S \cap T = \emptyset$. Two special classes of games are of special interest. The game (N, v) is convex if $v(S \cup T) + v(S \cap T) \geq v(S) + v(T)$ for all groups S and T in N. The game (N, v) is $[n, n\text{-}1]$ if $v(S) = 0$ for all $|S| \leq n-2$; in the following, we will use the notation $a_i = v(N - \{i\})$ for $i \in N$ and $a_0 = v(N)$.

Define the vector $x = (x_1, x_2, x_3, \ldots x_n)$ with real components to be an allocation of an n-person game where x_j is the value being allocated to player j. An allocation method or value is a function θ which, given any game (N, v), assigns an allocation $x = \theta(N, v)$ such that $\sum_{i \in N} x_i = v(N)$. This latter condition is referred to as efficiency. A value satisfies the equal treatment property if two individuals receive the same payoff whenever they have the same effect on the value function: if $v(S \cup \{i\}) = v(S \cup \{j\})$ for all $S \subseteq N - \{i, j\}$, then $\theta_i(N, v) = \theta_j(N, v)$. The core of a game (N, v), denoted by core(N, v), is the set of all efficient allocations such that $\sum_{i \in N} x_i \geq v(S)$, for all $S \subset N$. An allocation method, θ, is said to be group rational if whenever core(N, v) $\neq \emptyset$, it follows that $\theta(N, v) \in$ core(N, v).

We now define the monotonicity properties of interest. Let 1_S be the characteristic vector on the subset S of N, that is, the i-th component of 1_S is 1 if $i \in S$ and is 0 if $i \notin S$. The value θ is aggregate monotone at the game (N, v) if $\forall\, i \in N$ $\theta_i(N, v + \epsilon 1_N)$ is a nondecreasing function of ϵ in a neighborhood of $\epsilon = 0$. The value θ is group monotone at the game (N, v) if $\forall\, S \subseteq N$, $\forall\, i \in S$, $\theta_i(N, v + \epsilon 1_S)$ is a nondecreasing function of ϵ in a neighborhood of $\epsilon = 0$. The value θ is total population monotone at the game (N , v) if $\theta_i(S, v) \leq \theta_i(T, v)$ for all $i \in S \subseteq T \subseteq N$. Let TPM($\theta$) denote the set of games (N, v) at which the value θ is TPM. A value θ will be called Maximally TPM if there exists no value θ' for which TPM(θ) is a strict subset of TPM(θ').

III. METHODS:

We now define three values that are well known in the game theory literature and a new value. The Shapley Value (Shapley, 1953) for individual i is the average of the marginal values individual i brings to the group over all possible orderings:

$$\phi_i(N, v) = \sum \frac{(s-1)!\,(n-s)!}{n!} [\, v(S) - v(S - \{i\}) \,] \text{, where } s \text{ denotes } |S|.$$

Young (1985a) showed that the Shapley Value is the unique symmetric and strongly monotonic value, and so is group monotone on all games. Moulin (1988) states in an exercise that the Shapley Value is totally population monotone on convex games. It is also well-known that the Shapley Value is group rational on convex games but not on all games which possess a nonempty core.

Given a game (N, v), let $e(x, S) = v(S) - \sum_{i \in S} x_i$ be the excess of group S relative to the cost allocation x; this is a measure of how much group S is likely to complain about the allocation x, because e(x, S) is the difference between what group S can obtain on its own and what it would obtain according to x. Let e(x) be the vector of excesses e(x, S), $S \neq \emptyset, N$, ordered from highest to lowest. The nucleolus (Schmeidler, 1969) is the individually rational allocation $\nu(N, v)$ that minimizes e(x) lexicographically. In words, the nucleolus is the individually rational allocation which lexicographically minimizes the maximum excesses. Jew (1988) gave a sufficient for the nucleolus to be not group monotonic (with respect to coalitions other than N). Megiddo (1974) showed that the nucleolus is not aggregate monotone at a nine player game. Young (1985b) further showed that the nucleolus is not aggregate monotone at a six player game. However, the nucleolus is group rational.

We will need a characterization of the nucleolus due to Kohlberg (1971) in the third section. A collection $\mathfrak{B}$ of subsets of N is balanced if there exists $\lambda_S > 0$ such that $\sum_{S \in \mathfrak{B}} \lambda_S 1_S = 1_N$. An excess coalition array for an allocation x with respect to the game (N, v) is a partition $\mathfrak{B}_1, \ldots, \mathfrak{B}_q$ of $2^N - \{\emptyset, N\}$ satisfying $e(x, S) \geq e(x, T)$ whenever $S \in \mathfrak{B}_i$, $T \in \mathfrak{B}_j$ and $i \leq j$. An excess coalition array is balanced if $\bigcup_{i=1}^{p} \mathfrak{B}_i$ is balanced for $p = 1, \ldots, q$. Kolhberg (1971) proves that the allocation x is the nucleolus for the superadditive game (N, v) if and only if there exists a balanced excess coalition array for x with respect to (N, v). A nondegenerate game is a game for which the nucleolus yields a unique, balanced excess coalition array.

Given that the maximum an individual can expect is their separable value, $M_i = v(N) - v(N-\{i\})$, and the minimum is the maximum of what they can make with a group, $m_i = \max\{v(S) - \sum_{j \in S-\{i\}} M_j : i \in S \subseteq N\}$, the Tau value (Tijs, 1981) is the individually rational allocation that yields a straight-line compromise between the maximum and minimum entitlement allocations:

$$\tau(N, v) = \lambda m + (1 - \lambda) M$$

where

$$\lambda = \frac{\sum_{i=1}^{n} M_i - v(N)}{\sum_{i=1}^{n} M_i - \sum_{i=1}^{n} m_i}$$

This defines the Tau value on quasibalanced games. That is, games for which $m_i \leq M_i$ for all $i \in N$ and $\sum_{i \in N} m_i \leq v(N) \leq \sum_{i \in N} M_i$.

Define a new method, the <u>Maximum</u> <u>Egalitarian</u> <u>Method</u>, η, such that $\eta_i = Z_i(N, v) + \frac{1}{n}[v(N) - \sum_{j \in N} Z_j(N, v)]$ where $Z_i(N, v) = \max \{ \eta_i(N - k, v) \mid k \in N - i \}$.

A fundamental trade-off appears between the Shapley Value, which is group monotone but not group rational on all games, and the nucleolus, which is group rational but not group monotone on all games. Young (1985a) showed that no value is group rational and group monotone on games with five or more players. Jew and Housman (1989) showed that no value is group rational and group monotone on games with four or more players, but that there exists an infinite class of values that are group rational and group monotone on three player games. The primary motivation for this research is to discover the extent of this trade-off. For example, it is well-known that the per capita nucleolus is group rational and aggregate monotone. Returning to the nucleolus, we ask on which games this value is group monotone. A similar question for the Shapley Value would be on which games it is group rational. We have considered such questions with respect to other values and the total population monotonicity property also.

*** SECTION TWO ***

THE MOST TPM OF ALL

IA. One of the measures of fairness is TPM. I will exhibit a game for which three popular methods, ϕ, τ, ν, are not TPM, but for which a fourth method, η , is TPM.

Let $N = \{1, 2, 3\}$, $v(N) = 1$, $v(12) = .25$, $v(13) = v(23) = .666$, and $v(i) = 0 \ \forall \ i \in N$.

A. $\phi(N, v) = (.264, .264, .472)$. ϕ is not TPM at (N, v) because player 2 can make more in coalition $\{2, 3\}$ since $\phi(23, v) = (-, .333, .333)$ and .264 is not greater than .333.

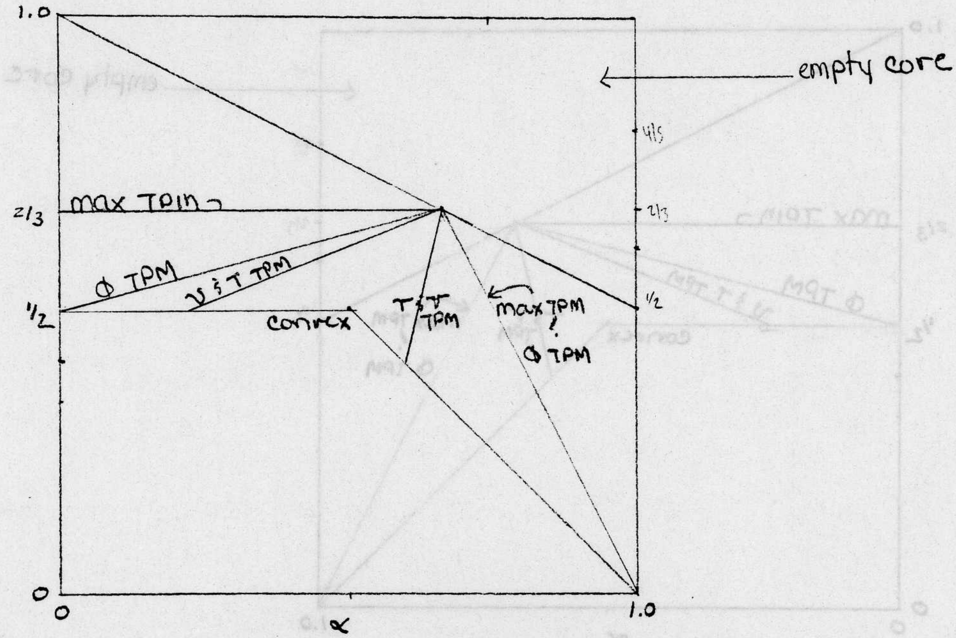
B. $\tau(N, v) = (.205, .205, .590)$. Likewise, τ is not TPM because player two can make more in the $\{2, 3\}$ coalition in which it will get .333.

C. $\nu(N, v) = (.195, .195, .611)$. Again, player 2 is better off in the $\{2, 3\}$ coalition as it will make .333 which is greater than .195.

D. $\eta(N, v) = (.333, .333, .333)$ which is TPM since no player can do better than .333 in any other coalition.

Thus, where ϕ, τ, ν fail to be TPM , η is TPM, suggesting it is in some sense "fairer."

I B. The following diagram geometrically represents the regions where each value is TPM. We consider three player games where $v(i) = 0 \ \forall \ i = 1, 2, 3$, $v(12) = \alpha$, $v(13) = v(23) = \beta$, $v(N) = 1$. The key idea is that for any symmetric method θ, θ is TPM at (N, v) if and only if $\theta_1(N, v) \geq \alpha/2, \beta/2$, $\theta_2(N, v) \geq \alpha/2, \beta/2$, and $\theta_3(N, v) \geq \beta/2$.



II. THEOREM: η is TPM on all [n, n-1] games for which any method satisfying the equal treatment property can be TPM on.

PROOF: The theorem will be shown by a series of three claims.

A. Suppose θ is any value satisfying the equal treatment property (ETP). Then $(N, v) \in \mathrm{TPM}(\theta) \Leftrightarrow \theta_i(N, v) \geq \max\{ \frac{a_j}{n-1} \mid j \neq i \}\ \forall i \in N$. Indeed,

$$(N, v) \in \mathrm{TPM}(\theta) \Leftrightarrow \forall i \in S \subseteq N \quad \theta_i(S, v) \leq \theta_j(T, v) \quad \text{by definition}$$

$$\Leftrightarrow \forall i \neq j \quad \theta_i(N\text{-}j, v) \leq \theta_i(N, v) \quad \text{since by ETP } \theta_i(S, v) = 0 \text{ if } |S| \leq n\text{-}2$$

$$\Leftrightarrow \forall i \neq j \quad \frac{a_j}{n-1} \leq \theta_i(N, v) \quad \text{since by ETP } \theta_i(N\text{-}j, v) = \frac{a_j}{n-1}$$

$$\Leftrightarrow \forall i \quad \max\{ \frac{a_j}{n-1} \mid j \neq i \} \leq \theta_i(N, v).$$

B. $(N, v) \in \mathrm{TPM}(\eta) \Leftrightarrow \sum_{i \in N} \max\{ a_j \mid j \neq i\} \leq (n-1)\, v(N)$. Indeed,

$$(N, v) \in \mathrm{TPM}(\eta) \Leftrightarrow \eta_i(N, v) \geq \max\{ \frac{a_j}{n-1} \mid j \neq i\} \quad \forall i \in N \text{ by A.}$$

$$\Leftrightarrow \max\{ \frac{a_j}{n-1} \mid j \neq i\} + \tfrac{1}{n}[\, v(N) - \sum_{i \in N} \max\{ \frac{a_j}{n-1} \mid j \neq i\}\,]$$

$$\geq \max\{ \frac{a_j}{n-1} \mid j \neq i\} \quad \forall i \in N \text{ by def. of } \eta$$

$$\Leftrightarrow v(N) \geq \sum_{i \in N} \max\{ \frac{a_j}{n-1} | j \neq i\}$$
$$\Leftrightarrow (n-1)v(N) \geq \sum_{i \in N} \max\{ a_j | j \neq i\} .$$

C. $\exists\ \theta$ such that θ TPM at (N, v) $\Rightarrow$ η is TPM at (N, v). Indeed, suppose θ is a value and (N, v) is a game.

$$(N, v) \in TPM(\theta) \Leftrightarrow \theta_i(N, v) \geq \max\{ \frac{a_j}{n-1} | j \neq i\} \quad \forall\ i \in N \qquad \text{by A.}$$
$$\Rightarrow v(N) = \sum_{i \in N} \theta_i(N, v) \geq \sum_{i \in N} \max\{ \frac{a_j}{n-1} | j \neq i\}$$
$$\Rightarrow (n-1)v(N) \geq \sum_{i \in N} \max\{ a_j | j \neq i\}$$
$$\Rightarrow (N, v) \in TPM(\eta) \qquad \text{by B.}$$

A, B, C $\Rightarrow$ η is TPM on all [n, n-1] games for which any method satisfying ETP can be TPM.

III. THEOREM: There is no value which is TPM on all games for which some value is TPM.

PROOF: Let $H = \{(N, v) : \exists\ \hat{\theta}$ such that $\hat{\theta}$ is TPM at $(N, v)\}$. Suppose θ is a value such that $TPM(\theta) = H$. Consider the four-player game (N, v) defined by $v(N) = 4$, $v(ijk) = 3$, $v(12) = 2$, and $v(S) = 0$ otherwise. Let $x = \theta(N, v)$. By efficiency, $x_1 + x_2 + x_3 + x_4 = 4$. By TPM applied to the three player coalitions, $x_i + x_j + x_k \geq 3$ for all three player coalitions $\{ijk\}$. Summing these inequalities and using efficiency, we obtain $4 \cdot 3 \leq 3 \cdot (x_1 + x_2 + x_3 + x_4) = 3 \cdot 4$. Hence, the inequalities must hold with equality. So, $\theta(N, v) = (1, 1, 1, 1)$ and $\theta(123, v) = (1, 1, 1)$.

Now, if we change the game slightly, we can show that this method is not TPM. Define (N, v) such that $v'(N) = 3.6$, $v'(134) = v'(234) = 0$, and leave all other coalitions as is. Since there is no change in the subgames determined by the coalitions {123} and {124}, players must receive the same allocation as before. That is, $\theta(123, v') = (1, 1, 1, -)$ and $\theta(124, v') = (1, 1, -, 1)$. In order for $(N, v') \in TPM(\theta)$, the grand coalition must consider the triples, and $\theta(N, v') \geq (1, 1, 1, 1)$. Yet, $\sum_{i \in N} \theta_i (N, v') > 3.6 = v(N)$, violating efficiency. Thus, θ is not TPM at (N, v').

Could this mean that (N, v$'$) is not in H? We will show that (N, v$'$) $\in$ TPM($\hat{\theta}$) for some other value $\hat{\theta}$. We need only define $\hat{\theta}$ on (N, v$'$) and its subgames. This method gives all allocations to players 1 and 2 . $\hat{\theta}$(12, v$'$) = (1, 1, –, –), and $\hat{\theta}$(123, v$'$) = $\hat{\theta}$(124, v$'$)= (1.5, 1.5, –, –). For the grand coalition, again allocate to 1 and 2 such that $\hat{\theta}$(1234, v$'$) = (1.8, 1.8, 0, 0). This method $\hat{\theta}$ is TPM at (N, v$'$) where θ was not. Yes, (N, v$'$) $\in$ H. We have shown that there is no value which is TPM for every game in H.

IV. Characterize the class of maximally TPM values.

LEMMA: θ is TPM at (N, v) $\Leftrightarrow$ θ_i(T-j, v) $\leq$ θ_i(T, v) $\forall$ i, j $\in$ T $\subseteq$ N

PROOF: Given θ is TPM @ (N, v), show θ_i(T-j, v) $\leq$ θ_i(T, v) $\forall$ i,j $\in$ T $\subseteq$ N .

By definition of TPM, θ_i(S, v) $\leq$ θ_i(T, v) $\forall$ i,j $\in$ S $\subseteq$ T $\subseteq$ N. Therefore, let S = T - j, and consequently θ_i(T-j, v) $\leq$ θ_i(T, v) $\forall$ i,j $\in$ T $\subseteq$ N .

Given θ_i(T-j, v) $\leq$ θ_i(T, v) $\forall$ i,j $\in$ T $\subseteq$ N , show θ is TPM @ (N, v). Suppose we only have one and two player coalitions. For this game to be TPM θ_i(i, v) $\leq$ θ_i(ij, v) $\forall$ i,j $\in$ T $\subseteq$ N . Now, if we change the game by adding a three person coalition, for TPM to hold we must have (a) θ_i(i, v) $\leq$ θ_i(ij,v) , (b) θ_i(i, v) $\leq$ θ_i(ijk, v) , and (c) θ_i(ij, v) $\leq$ θ_i(ijk, v) $\forall$ i,j $\in$ T $\subseteq$ N . By given, both a and c are true. This, in turn, forces b to be true, and hence TPM on this game. If we extend this process we can inductively see that if all the allocations of a subset are greater than or equal to all the possible allocation of that subset minus one player, then the value will have to be TPM. That is, θ_i(T-j, v) $\leq$ θ_i(T, v) $\forall$ i,j $\in$ T $\subseteq$ N $\Rightarrow$ θ is TPM @ (N, v). Hence, θ is TPM @ (N, v) $\Leftrightarrow$ θ_i(T-j, v) $\leq$ θ_i(T, v) $\forall$ i,j $\in$ T $\subseteq$ N .

We now note that given any value θ, there exists a semi-value α for which $\sum_{i \in N} \alpha_i(N, v) = 1$ for all games (N, v) and θ can be written as

$$\theta_i(N, v) = z_i(N, v) + \alpha_i(N, v)[\, v(N) - \sum_{j \in N} z_j(N, v)\,]$$

where

$$z_j(N, v) = \max \{ \theta_j(N-k, v) \mid k \in N-j \}.$$

The maximally TPM values can now be characterized as those values for which α satisfies a seemingly natural property.

THEOREM: A value θ is maximally TPM $\Leftrightarrow$ $\alpha(N, v) \geq 0$ for all games (N, v) satisfying $\sum_{j \in N} z_j(N, v) < v(N)$.

PROOF:

A. Given θ is a maximally TPM value, we will show that $\alpha(N, v) \geq 0$ whenever (N, v) satisfies $\sum_{j \in N} z_j(N, v) < v(N)$. Indeed, suppose that this were not true for θ, that is, $\exists$ (N, v) such that $\sum_{j \in N} z_j(N, v) < v(N)$ but $\alpha_i(N, v) < 0$ for some $i \in N$. We can see that θ is not TPM at (N, v) because $\theta_i(N, v) < z_i(N, v) = \theta_i(N-k, v)$ for some $k \in N-i$. So, we define a new value $\hat{\theta}$ to show that θ is not maximally TPM. Let $\hat{\theta}(\tilde{N}, \tilde{v}) = \theta(N, v)$ for all $(\tilde{N}, \tilde{v}) \neq (N, v)$, and $\hat{\theta}(N, v) = z_i(N, v) + \frac{1}{n}[\, v(N) - \sum_{j \in N} z_j(N, v)\,]$. Now, suppose $(M, u) \in TPM(\theta)$, and $i, j \in S \subseteq T \subseteq M$. Then (S, u), $(T, u) \in TPM(\theta) \Rightarrow (S, u)$, $(T, u) \neq (N, v)$ since θ is not TPM at (N, v). Hence, $\hat{\theta}_i(S, u) = \theta_i(S, u) \leq \theta_i(T, u) = \hat{\theta}_i(T, u) \Rightarrow (M, u) \in TPM(\hat{\theta})$, which shows that $\hat{\theta}$ is TPM wherever θ is TPM. Moreover, by definition of $\hat{\theta}(N, v)$, $\hat{\theta}_i(N, v) \geq \hat{\theta}_i(N-k, v)$ $\forall\, k \in N-i$ which shows that $\hat{\theta}$ is TPM in more places than θ. Thus, when $\alpha(N, v) < 0$, θ is not maximally TPM. By the contrapositive, for θ maximally TPM, $\alpha(N, v) \geq 0$ whenever (N, v) satisfies $\sum_{j \in N} z_j(N, v) < v(N)$.

B. Suppose θ is a value for which $\alpha(N, v) \geq 0$ for all games (N, v) satisfying $\sum_{j \in N} z_j(N, v) < v(N)$. Suppose also that $\hat{\theta}$ is a value which satisfies $TPM(\theta) \subseteq TPM(\hat{\theta})$. We will show that θ is maximally TPM through a series of claims that eventually show that $TPM(\theta) = TPM(\hat{\theta})$.

B.1. $(N, v) \in TPM(\theta) \Leftrightarrow \sum_{j\in N} z_j(N, v) \leq v(N)$.

B.1. i. Given $\sum_{j\in N} z_j(N, v) \leq v(N)$, show $(N, v) \in TPM(\theta)$. Suppose θ were not TPM. Then $\exists\, i \neq k$ for which $z_i(N, v) + \alpha_i(N, v)\,[\, v(N) - \sum_{j\in N} z_j(N, v)\,] = \theta_i(N, v) < \theta_i(N-k, v) \leq z_i(N, v)$. Therefore, $\alpha_i(N, v)[\, v(N) - \sum_{j\in N} z_j(N, v)\,] < 0$. Since $\sum_{j\in N} z_j(N, v) \leq v(N)$ by the given for i, $\alpha_i(N, v) < 0$ which contradicts the property of θ given in the first sentence of B.

B.1. ii. Given $(N, v) \in TPM(\theta)$, show $\sum_{j\in N} z_j(N, v) \leq v(N)$. Since $\sum_{j\in N} \alpha_j(N, v) = 1$, $\exists\, i \in N$ such that $\alpha_i(N, v) > 0$. For this i, $z_i(N, v) + \alpha_i(n, v)[\, v(N) - \sum_{j\in N} z_j(N, v)\,] = \theta_i(N, v) \geq z_i(N, v)$ since $(N, v) \in TPM(\theta)$. This implies that $\alpha_i(N, v)[\, v(N) - \sum_{j\in N} z_j(N, v)\,] \geq 0$ which implies that $v(N) - \sum_{j\in N} z_j(N, v) \geq 0$ since $\alpha_i(N, v) > 0$. So, $\sum_{j\in N} z_j(N, v) \leq v(N)$.

B. 2. $\hat{\theta}(N, v) = \theta(N, v)$ whenever $(N, v) \in TPM(\theta)$. We show this by induction on the number of players. Clearly, $\hat{\theta}(N, v) = \theta(N, v)$ whenever $|N| = 1$. Now suppose that the two values are equal on all games with fewer than $n-1$ players, and let $(N-n, v) \in TPM(\theta)$ be a game on $n-1$ players. We will show that $\theta(N-n, v) = \hat{\theta}(N-n, v)$, which will complete the induction. For each $i \in N-n$, define a new game (N, v^i) as follows:

$$v^i(S) = \begin{cases} v(S) & \text{if } n \notin S \\ v(S-n+i) & \text{if } n \in S \text{ and } i \notin S \\ v(S-n) + \theta_i(S-n, v) & \text{if } n \in S \text{ and } i \in S \end{cases}$$

We claim that for all $j \in S \subseteq N$:

$$\theta_j(S, v^i) = \begin{cases} \theta_j(S, v) & \text{if } n \notin S \\ \theta_j(S-n+i, v) & \text{if } n \in S,\ i \notin S, \text{ and } j \neq n \\ \theta_i(S-n+i, v) & \text{if } n \in S,\ i \notin S, \text{ and } j = n \\ \theta_j(S-n, v) & \text{if } n \in S,\ i \in S, \text{ and } j \neq n \\ \theta_i(S-n, v) & \text{if } n \in S,\ i \in S, \text{ and } j = n \end{cases}$$

Indeed, if $n \notin S$, then $(S, v^i) = (S-n, v)$. If $n \in S$ and $i \notin S$, then $(S, v^i) = (S-n+i, v)$

with the understanding that player n in (S, v^i) corresponds to player i in $(S-n+i, v)$. So, the only cases for which the claim in not clearly true is for S containing both i and n. We show the claim inductively on the size of S. Suppose the claim holds for all proper subsets of S, and $i, n \in S$. If $j \in S-n$, then

$$\begin{aligned} z_j(S, v^i) &= \max\{ \theta_j(S\text{-}k, v^i) \mid k \in S\text{-}j \} \\ &= \max\{ \theta_j(S\text{-}n, v^i), \theta_j(S\text{-}i, v^i), \theta_j(S\text{-}k, v^i) \mid k \in S - \{j, n, i\} \} \\ &= \max\{ \theta_j(S\text{-}n, v), \theta_j(S\text{-}k\text{-}n, v) \mid k \in S - \{j, n, i\} \} \text{ by induction hypothesis} \\ &= \theta_j(S\text{-}n, v) \text{ since } (N\text{-}n, v) \in TPM(\theta). \end{aligned}$$

If $j=n$, then

$$\begin{aligned} z_n(S, v^i) &= \max\{ \theta_n(S\text{-}k, v^i) \mid k \in S\text{-}n \} \\ &= \max\{ \theta_n(S\text{-}i, v^i), \theta_j(S\text{-}k, v^i) \mid k \in S - \{n, i\} \} \\ &= \max\{ \theta_i(S\text{-}n, v), \theta_i(S\text{-}n\text{-}k, v) \mid k \in S - \{n, i\} \} \text{ by induction hypothesis} \\ &= \theta_i(S\text{-}n, v) \text{ since } (N\text{-}n, v) \in TPM(\theta). \end{aligned}$$

Now, by summing all the z_i's, we see that

$$\begin{aligned} \sum_{j \in S} z_j(S, v^i) &= \sum_{j \in S-n} \theta_j(S\text{-}n, v) + \theta_i(S\text{-}n, v) \\ &= v(S\text{-}n) + \theta_i(S\text{-}n, v) \\ &= v^i(S). \end{aligned}$$

Hence, $\theta_j(S, v^i) = z_j(S, v^i)$ for all $j \in S$ and our claim follows by induction. That is, $\theta_j(S, v^i) = \theta_j(S\text{-}n, v)$ for all $j \in S - n$ and $\theta_n(S, v^i) = \theta_i(S\text{-}n, v)$. It is now straight-forward to show that $(N, v^i) \in TPM(\theta)$ which then implies that $(N, v^i) \in TPM(\hat{\theta})$. So,

$$\begin{aligned} \hat{\theta}_i(N\text{-}n, v) &= \hat{\theta}_i(N\text{-}n, v^i) \text{ since } (N\text{-}n, v) = (N\text{-}n, v^i) \\ &= \hat{\theta}_n(N\text{-}i, v^i) \text{ since } (N\text{-}n, v^i) = (N\text{-}i, v^i) \text{ with i and n exchanged} \\ &\leq \hat{\theta}_n(N, v^i) \quad \text{by TPM} \\ &= \sum_{j \in N} \hat{\theta}_j(N, v^i) - \sum_{j \in N\text{-}n} \hat{\theta}_j(N, v^i) \\ &\leq v^i(N) - \sum_{j \in N\text{-}n} \hat{\theta}_j(N\text{-}n, v^i) \qquad \text{by efficiency and TPM} \end{aligned}$$

$$= v^i(N) - v^i(N\text{-}n) \quad \text{by efficiency}$$

$$= v^i(N) - v(N\text{-}n) \quad \text{since } (N\text{-}n, v^i) = (N\text{-}n, v)$$

$$= \sum_{j \in N} \theta_j(N, v^i) - \sum_{j \in N\text{-}n} \theta_j(N\text{-}n, v) \quad \text{by efficiency}$$

$$= \sum_{j \in N\text{-}n} \theta_j(N\text{-}n, v) + \theta_i(N\text{-}n, v) - \sum_{j \in N\text{-}n} \theta_j(N\text{-}n, v) \quad \text{by claim formulas}$$

$$= \theta_i(N\text{-}n, v).$$

This shows us that $\sum_{i \in N\text{-}n} \hat{\theta}_i(N\text{-}n, v) \leq \sum_{i \in N\text{-}n} \theta_i(N\text{-}n, v)$. However, since each of these terms is equal to $v(N\text{-}n)$, all of the proceeding inequalities must be equalities. Hence, $\hat{\theta}_i(N\text{-}n, v) = \theta_i(N\text{-}n, v)$ which is what we were to show.

B. 3. If $(N, v) \notin TPM(\theta)$, then $(N, v) \notin TPM(\hat{\theta})$. Indeed, assume $(N, v) \notin TPM(\theta)$, and that θ is TPM at all subgames of (N, v). From part B.1., this implies that $\sum_{j \in N} \max\{\theta_j(N-k, v) \mid k \in N\text{-}j\} = \sum_{j \in N} z_j(N, v) > v(N)$. Since θ is TPM at all subgames of (N, v), part B.2. implies that $\sum_{j \in N} \max\{\hat{\theta}_j(N\text{-}k, v) \mid k \in N\text{-}j\} = \sum_{j \in N} \max\{\theta_j(N\text{-}k, v) \mid k \in N\text{-}j\} >$ $v(N)$. However, by TPM and efficiency, $v(N) = \sum_{j \in N} \hat{\theta}_j(N, v) \geq \sum_{j \in N} \max\{\hat{\theta}_j(N\text{-}k, v) \mid k \in N\text{-}j\} >$ $v(N)$. This leads to a contradiction since $v(N)$ is not strictly greater than itself. By this we can say $\hat{\theta}$ is not TPM on (N, v). Hence, $(N, v) \notin TPM(\theta) \Rightarrow (N, v) \notin TPM(\hat{\theta})$.

By combining parts A., B.1., B.2., and B.3., we prove our theorem.

*** SECTION THREE ***

AGGREGATE AND GROUP MONOTONICITY AS SEEN BY THE NUCLEOLUS

I. It has been previously proven that the nucleolus is not aggregate monotonic for games with six or more players, and is aggregate monotonic for all three player games. We will now show by counter examples that the nucleolus is not aggregate monotonic for all four and five player games.

A. Example 1. Consider the following 4 player game: $v(S) = 0$ for all S except $v(123) = 4$; $v(124) = 7$; $v(234) = 8$; and $v(1234) = 10.5 + \epsilon$. The nucleolus is $(1.25 + 0.5\epsilon, 4.25 - 0.5\epsilon, 1.75 + 0.5\epsilon, 3.25 + 0.5\epsilon)$.

Why is this the nucleolus? Let $z = (1.25 + 0.5\epsilon, 4.25 - 0.5\epsilon, 1.75 + 0.5\epsilon, 3.25 + 0.5\epsilon)$.
First, collect coalitions by grouping them in terms of their excess vectors, in decreasing order.
$\mathfrak{B}_1 = \{234, 1\}$; $\mathfrak{B}_2 = \{124, 3\}$; $\mathfrak{B}_3 = \{13\}$; $\mathfrak{B}_4 = \{4, 123\}$; $\mathfrak{B}_5 = \{2\}$; ...
By this we mean that $e(z, 1) = e(z, 234) > e(z, 124) = e(z, 3) > e(z, 13)$ .... Suppose x is the nucleolus. Since x is the imputation that minimizes the maximum excess and z is an imputation with a maximum excess of $e(z, 1) = -1.25 - 0.5\epsilon$, no excess of x can exceed $e(z, 1)$.
In particular,

$$e(x, 1) = v(1) - x_1 = 0 - x_1 \leq -1.25 - 0.5\epsilon.$$

$$e(x, 234) = v(234) - x_2 - x_3 - x_4 = 8 - x_2 - x_3 - x_4 \leq -1.25 - 0.5\epsilon.$$

Combining the two inequalities, we obtain $8 - x_1 - x_2 - x_3 - x_4 = 8 - v(N) = 8 - 10.5 - \epsilon \leq -2.5 - \epsilon$. Simplifying , we get $0 \leq 0$. This means the previous inequality should be a strict equality. In other words, $-x_1 = -1.25 - 0.5\epsilon$ or $x_1 = 1.25 + 0.5\epsilon$, and $e(x, 1) = e(x, 234) = e(z, 1)$. So, x and z have the same maximum excess which is obtained on the same number of coalitions. Now since x is the imputation that lexicographically minimizes the maximum excesses, no excess other than $e(x, 1)$ and $e(x, 234)$ can exceed the second highest excess of z. Specifically, $e(x, 3) \leq e(z, 3)$ and $e(x, 124) \leq e(z, 3)$. Now an argument similar to the one given earlier yields $x_3 = 1.75 + 0.5\epsilon$, $e(x, 3) = e(x, 124) = e(z, 3)$. In a like manner we determine x_4 from $\mathfrak{B}_4$, where $x_4 = 3.25 + 0.5\epsilon$. We know from efficiency $v(N) = x_1 + x_2 + x_3 + x_4$. Using known values for $v(N)$, x_1, x_3, x_4, we can solve for $x_2 = 4.25 - 0.5\epsilon$. Therefore, $x = (1.25 + 0.5\epsilon, 4.25 - 0.5\epsilon, 1.75 + 0.5\epsilon, 3.25 + 0.5\epsilon)$ is the nucleolus. Notice, as we increase $v(N)$ by increasing ϵ, the value of x_2 decreases. By definition, the nucleolus is not aggregate monotonic. If we change the game so that $v(N) \geq 13$ and the rest remains the same, the nucleolus for those games will be aggregate monotonic.

B. <u>Example</u> <u>2.</u> Consider the following five player game: $v(S) = 0$ for all S in N except for $v(1234) = 25$; $v(2345) = 10$; $v(1345) = 20$; $v(1245) = 45$; $v(1235) = 30$; $v(12345) = 49 + \epsilon$, for some $\epsilon > 0$. Using similar methods to example 1, the nucleolus for this game can be found to be $x = (12.75 - 0.25\epsilon, 12.75 - 0.25\epsilon, 2 + 0.5\epsilon, 9.5 + 0.5\epsilon, 12.0 + 0.5\epsilon)$ where the excess vectors form the following sets: $\mathfrak{B}_1 = \{1245, 3\}$; $\mathfrak{B}_2 = \{4, 1235\}$; $\mathfrak{B}_3 = \{34\}$; $\mathfrak{B}_4 = \{5, 1234\}$; $\mathfrak{B}_5 = \{1, 2\}$ .... Here, x_1 and x_2 will decrease as $v(N)$ is increased by some sufficiently small ϵ. Hence, this game is not aggregate monotonic. However, if we change the game, by keeping all values the same except for $v(N)$ which is made greater than or equal to 50, the new game will be aggregate monotonic. For example, define a new game w such that $w(S) = v(S)$ for all S in N except when $S = N$, where $w(N) = 50 + \epsilon$ for some $\epsilon > 0$. The nucleolus for w is $x_w = (12.5 + \epsilon, 12.5 + \epsilon, 2.5 + 0.5\epsilon, 10.0 + 0.5\epsilon, 12.5 + 0.5\epsilon)$.

II. THEOREM: The nucleolus is aggregate monotonic at the nondegenerate, [n, n-1] game (N, v) if and only if there is no three player coalition R that satisfies $e(x, i) = e(x, i^c) > e(x, j^c), e(x, j)$ for all $i \in R$ and $j \notin R$ where x is the nucleolus of (N, v).

PROOF:

A. Assume there exists a three player coalition R that satisfies $e(x, i) = e(x, i^c) > e(x, j^c)$, $e(x, j)$ for all $i \in R$ and $j \notin R$ where x is the nucleolus of (N, v). Without loss of generality, we will denote the three player coalition as $\{1, 2, 3\}$ for clarity. From the definition of excess vectors, $v(1) - x_1 = a_1 - x_2 \ldots - x_n$. This can be rewritten as $x_1 = [v(N) - a_1]/2$. Similarly, $x_2 = [v(N) - a_2]/2$, and $x_3 = [v(N) - a_3]/2$. Since the games we are looking at are efficient, $v(N) = \sum_{i \in N} x_i$. Substituting the above values, we obtain $v(N) = ([v(N) - a_1]/2) + ([v(N) - a_2]/2) + ([v(N) - a_3]/2) + \sum_{j=4}^{n} x_j$, or $\sum_{j=4}^{n} x_j = [-v(N) + a_1 + a_2 + a_3]/2$. This means there exists at least one payoff between x_4 and x_n that decreases as $v(N)$ increases. Hence, the nucleolus is not aggregate monotonic.

B. Assume there is no three player coalition R that satisfies $e(x, i) = e(x, i^c) > e(x, j^c)$, $e(x, j)$ for all $i \in R$ and $j \notin R$ where x is the nucleolus of (N, v). We want to find the maximum excesses to solve for possible excess coalition arrays. Let $S =$ coalition consisting of q elements, where $q < n$. If $q = 1$, then $e(x, S) = e(x, i) = v(i) - x_i = 0 - x_i$. This could be a maximum. If $2 \leq q \leq n - 2$ then $e(x, S) = v(S) - \sum_{i \in S} x_i = 0 - \sum_{i \in S} x_i$. From above we can substitute $-x_i = e(x, i)$ to get $e(x, S) = \sum_{i \in S} e(x, i)$. Since $x_i \geq 0$ for all $i \in N$, $e(x, i) \leq 0$. Hence, $\sum_{i \in S} e(x, i) \leq e(x, j)$ for each $j \in S$. Therefore, each singleton subset of S will have a larger excess than $e(x, S)$ and $e(x, S)$ is not a maximum. If $q = n - 1$ then $e(x, S) = e(x, i^c) = a_i - \sum_{i \in S} x_j$. This could be a maximum. Using Kolhberg's Theorem, we know that the excess coaltion arrays must be balanced in order for x to be the nucleolus. Hence, the only essential cases are when S has either 1 or n-1 elements. Specifically, there are six possible cases:

case 1: $e(x, 1) = e(x, 1^c) > e(x, 2) = e(x, 2^c) > e(x, 3) = \ldots = e(x, n)$

case 2: $e(x, 1) = e(x, 1^c) > e(x, 2) = e(x, 2^c) > e(x, 3^c) = \ldots = e(x, n^c)$

case 3: $e(x, 1) = e(x, 1^c) > e(x, 2) = e(x, 3) = \ldots = e(x, n)$

case 4: $e(x, 1) = e(x, 1^c) > e(x, 2^c) = e(x, 3^c) = \ldots = e(x, n^c)$

case 5: $e(x, 1) = e(x, 2) = e(x, 3) = \ldots = e(x, n)$

case 6: $e(x, 1^c) = e(x, 2^c) = e(x, 3^c) = \ldots = e(x, n^c)$

Note that in cases 1 and 2, $e(x, 12)$ could be less than $e(x, 2)$ and greater than both $e(x, 3)$, and $e(x, 3^c)$, which would not change the formulas for the nucleolus to be given below. Consider first case 1:

$$e(x, 1) = e(x, 1^c),$$

$$e(x, 2) = e(x, 2^c), \text{ and}$$

$$e(x, 3) = \ldots = e(x, n)$$

which is equivalent to

$$v(1) - x_1 = a_1 - (a_0 - x_1),$$

$$v(2) - x_2 = a_2 - (a_0 - x_2), \text{ and}$$

$$v(3) - x_3 = \ldots = v(n) - x_n.$$

Solving for x, we obtain

$$x_1 = [a_0 - a_1 + v(1)]/2,$$

$$x_2 = [a_0 - a_2 + v(2)]/2, \text{ and}$$

$$x_j = [\{a_1 - v(1) + a_2 - v(2)\}/2 + (n-2)\, v(j) - \sum_{i=3}^{n} v(i)]/(n-2), \text{ for } 3 \leq j \leq n.$$

By incrementing $v(N) = a_0$ by ϵ, both x_1 and x_2 result in an increment by $\epsilon/2$, and $x_3, \ldots,$ x_n remain unchanged. Thus, the nucleolus is aggregate monotonic for games in this case. Similarly, we can solve case 3:

$$x_1 = [a_0 - a_1 + v(1)]/2, \text{ and}$$

$$x_j = [\{a_1 - v(1)\}/2 + (n-1)\, v(j) - \sum_{i=2}^{n} v(i)]/(n-1), \text{ for } 2 \leq j \leq n.$$

and case 5:

$$x_j = [n \cdot v(j) + a_0 - \sum_{i \in N} v(i)]/n, \text{ for } 1 \leq j \leq n.$$

In both cases we can see that the nucleolus is aggregate monotonic for games in the designated cases.

We now turn to case 2:

$$e(x, 1) = e(x, 1^c),$$

$$e(x, 2) = e(x, 2^c), \text{ and}$$

$$e(x, 3^c) = \ldots = e(x, n^c)$$

which is equivalent to

$$v(1) - x_1 = a_1 - (a_0 - x_1),$$

$$v(2) - x_2 = a_2 - (a_0 - x_2), \text{ and}$$

$$a_3 - x_3 = \ldots = a_n - x_n.$$

Solving for x, we obtain

$$x_1 = [a_0 - a_1 + v(1)]/2,$$

$$x_2 = [a_0 - a_2 + v(2)]/2, \text{ and}$$

$$x_j = [\{a_1 - v(1) + a_2 - v(2)\}/2 - (n-2)\, a_j + \sum_{i=3}^{n} a_i]/(n-2), \text{ for } 3 \le j \le n.$$

By incrementing $v(N) = a_0$ by ϵ, we obtain that both x_1 and x_2 increment by $\epsilon/2$, and x_3, ..., x_n remain unchanged. Thus, the nucleolus is aggregate monotonic for games in this case.

Similarly, we can solve case 4:

$$x_1 = [a_0 - a_1 + v(1)]/2, \text{ and}$$

$$x_j = [\{a_0 + a_1 - v(1)\}/2 - (n-1)\, a_j + \sum_{i=2}^{n} a_i]/(n-1), \text{ for } 2 \le j \le n$$

and case 6:

$$x_j = [a_0 - n\, a_j + \sum_{i \in S} a_i]/n, \text{ for } 1 \le j \le n.$$

In either case, an increment of a_0 by ϵ results in an increment of every x_j by some positive multiple of ϵ. Hence, the nucleolus is aggregate monotonic for games in these cases.

We have obtained formulas for the nucleolus for all possible cases when there is no three player coalition R that satisfies $e(x, i) = e(x, i^c) > e(x, j^c)$, $e(x, j)$ for all $i \in R$ and $j \notin R$ where x is the nucleolus of (N, v). In all of these cases, the nucleolus is aggregate monotonic for the specified games.

III. THEOREM: For $[n, n-1]$, nondegenerate games, the nucleolus is aggregate monotonic if and only if it is group monotonic.

PROOF:

A. Suppose the nucleolus is group monotonic on all $[n, n-1]$, nondegenerate games. By the definition of group monotonicity the nucleolus is aggregate monotonic.

B. Suppose the nucleolus is aggregate monotonic for all $[n, n-1]$, nondegenerate games. From the previous theorem, there is no three player coalition R that satisfies $e(x, i) = e(x, i^c) > e(x, j^c)$, $e(x, j)$ for all $i \in R$ and $j \notin R$ where x is the nucleolus of (N, v). This results in there being only six possible formulas for the nucleolus, as shown in the previous proof. We want to test for group monotonicity. To do this we increment $v(S)$ by some small ϵ while leaving all other values constant, for all $S \subseteq N$, and look at x_i for all $i \in S$. If they remain constant or are also incremented, then the nucleolus is group monotonic for all games in that case. In $[n, n-1]$ games, $v(i)$ and $v(i^c)$ for all $i \in N$ are the only values that need be incremented since the other coalitions do not affect x_i. For example, case 1:

$$x_1 = [a_0 - a_1 + v(1)]/2,$$

$$x_2 = [a_0 - a_2 + v(2)]/2, \text{ and}$$

$$x_j = [\{a_1 - v(1) + a_2 - v(2)\}/2 + (n-2)\, v(j) - \sum_{i=3}^{n} v(i)]/(n-2), \text{ for } 3 \leq j \leq n.$$

If we increment $v(1)$ by ϵ we need only to look at x_1, which in this case is incremented by $\epsilon/2$. If we increment a_1 by ϵ, we need to look at $x_2, \ldots, x_n$. Here x_2 increases by $\epsilon/2$, and the remaining x_j's for $3 \leq j \leq n$ hold at the same value. This process can be repeated for the remaining coalitions and it can be seen that this case is group monotonic. In general, we can look at the formulas for the nucleolus and look at the coefficients of the $v(S)$ being changed. If the coefficient is nonnegative for all x_i for all $i \in S$, then the case is group monotonic. Upon examination of the general formulas, as above, one can see that the nucleolus is group monotonic for all games in the six essential cases, when the nucleolus is aggregate monotonic. Therefore, for all $[n, n-1]$, nondegenerate games, the nucleolus is aggregate monotonic if and only if it is group monotonic.

IV. THEOREM: For any $[n, n-1]$, nondegenerate games (N, v), if (N, v) is convex, then the nucleolus is aggregate monotonic.

PROOF(by contrapostion): Assume the nucleolus is not aggregate monotonic. From a previously proven theorem, there exists a three player coalition R that satisfies $e(x, i) = e(x, i^c) > e(x, j^c)$, $e(x, j)$ for all $i \in R$ and $j \notin R$ where x is the nucleolus of (N, v). Without loss of generality, we will denote the three player coalition as $\{1, 2, 3\}$ for clarity. This implies $e(x, 1) = e(x, 1^c)$; $e(x, 2) = e(x, 2^c)$; $e(x, 3) = e(x, 3^c)$. These equations can be rewritten as

$$v(1) - x_1 = a_1 - x_2 \ldots - x_n$$

$$v(2) - x_2 = a_2 - x_1 - x_3 \ldots - x_n$$

$$v(3) - x_3 = a_3 - x_1 - x_2 - x_4 \ldots - x_n,$$

or

$$x_1 = (a_0 - a_1) / 2$$

$$x_2 = (a_0 - a_2) / 2$$

$$x_3 = (a_0 - a_3) / 2$$

Because the excess coalition arrays are arranged in decreasing order, $e(x, 3) > e(x, \{4, .., n\})$. By definitions of excess vectors and efficiency, $e(x, \{4, .., n\}) = x_1 + x_2 + x_3 - a_0$. Substituting in the inequality for $e(x, 3)$ and $e(x, \{4, .., n\})$, we obtain $(a_3 - a_0)/2 > [(3a_0 - a_1 - a_2 - a_3)/2] - a_0$. Simplifying, we obtain $a_1 + a_2 + 2a_3 > 2a_0$. If the game is convex, then it follows that $a_1 + a_3 \leq a_0$ and $a_1 + a_3 \leq a_0$. Summing these inequalities, we obtain $a_1 + a_2 + 2a_3 \leq 2a_0$. Since we showed that $a_1 + a_2 + 2a_3 > 2a_0$, it follows that the game is not convex.

V. THEOREM: For all $[n, n-1]$, nondegenerate games (N, v), if (N, v) is convex, then the nucleolus is group monotone.

The proof for this theorem directly follows from the above theorems.

VI. EXAMPLES:

The previous theorems may not hold at a degenerate game. We exhibit two games that are degenerate and convex. The nucleolus is aggregate monotone at the first and is not aggregate monotone at the second.

A. Consider the $[4, 3]$ games defined by $a_0 = 4 + 4\epsilon$ and $a_1 = a_2 = a_3 = a_4 = 2$. For all $\epsilon \geq -1/2$, the nucleolus is $(1 + \epsilon, 1 + \epsilon, 1 + \epsilon, 1 + \epsilon)$. At $\epsilon = 0$, the game is convex, degenerate, and the nucleolus is clearly aggregate monotonic.

B. Consider the $[4, 3]$ games defined by $a_0 = 4 + 4\epsilon$, $a_1 = a_2 = a_3 = 2$ and $a_4 = 0$. For all $\epsilon \in [-1/2, 0]$, the nucleolus is $(1 + 2\epsilon, 1 + 2\epsilon, 1 + 2\epsilon, 1 - 2\epsilon)$. For all $\epsilon \geq 0$, the nucleolus is $(1 + \epsilon, 1 + \epsilon, 1 + \epsilon, 1 + \epsilon)$. At $\epsilon = 0$, the game is convex, degenerate, and the nucleolus is clearly not aggregate monotonic.

REFERENCES

Brown, D., L. Jew, and D. Macdonald (1988), The name of the game, Summer REU project report.

Jew, L. and D. Housman (1989), Core and monotonic allocation procedures, manuscript.

Kohlberg, E. (1971), On the nucleolus of a characteristic function game, *SIAM SIAM Journal on Applied Mathematics* 20, 62-66.

Megiddo, N. (1974), On the nonmonotonicity o the bargaining set, the kernel, and the nucleolus of a game, *SIAM Journal on Applied Mathematics* 27, 355-358.

Moulin, H. (1988), *Axioms of Cooperative Decision Making*, Cambridge: Cambridge University Press.

Schmeidler, D. (1969), The nucleolus of a characteristic function game, *SIAM Journal on Applied Mathematics* 17, 1163-1170.

Shapley, L.S. (1953), A value for n-person games, in H. W. Kuhn and A. W. Tucker (Eds) *Contributions to the Theory of games II, Annals of Mathematics Studies No.* 28, Princeton, NJ: Princeton University Press, 307-317.

Tijs, S. H. (1981), Bounds for the core and the τ-value, in O. Moeschlin and D. Pallaschke (Eds) *Game Theory and Mathematical Economics*, Amsterdam: North-Holland Publishing Company, 123-132.

Young, H. P. (1985a), Monotonic solutions of cooperative games, *International Journal of Game Theory* 14, 65-72.

Young, H. P. (1985b), Cost allocation, in H. P. Young (Ed) *Fair Allocation: Proceedings of Symposia in Applied Mathematics*, Providence, RI: American Mathematical Society, 69-94.