# Values on Partition Function Form Games

Stephanie Merki
August 6, 1991

Partition function form games were first introduced in Lucas and Thrall in 1963 as a generalized form of characteristic function form games. Both R.B. Myerson and E.M. Bolger have defined values on partition function form games. In this paper an extension of the Shapley value is sought which will satisfy linearity, efficiency, symmetry and dummy. Different axioms are then incorporated to place bounds on the various remaining parameters.

The following background information, along with 12 axioms and 7 definitions, is key to the paper.

<u>Background Information</u>:

$N = \{1,2,\ldots,n\}$ is the set of players in a n-person game.

$CL = \{S \mid S \subseteq N, S \neq \emptyset\}$ is the set of coalitions of N.

PT = set of partitions of N: $\{S^1,\ldots,S^m\} \in PT$ iff

$S^1 \cup \ldots \cup S^m = N$, $\forall j\ S^j \neq \emptyset, \forall k\ S^k \cap S^j = \emptyset$ if $k \neq j$.

ECL = set of embedded coalitions: $\{(S,P) \mid S \in P \in PT\}$.

A game in partition function form is any $W \in R^{ECL}$.

$W(S;P)$ is the amount S would receive if partition P formed.

$\Phi(W)$ is a payoff vector for the game W.

$\Phi_i(W)$ is a payoff or allocation to player i on game W.

A game is w-superadditive if

$W(S;P)+W(T;P) \leq W(S \cup T; P-\{S,T\} \cup \{S \cup T\}) \forall (S;P),(T;P) \in ECL$.

A game is w-coalition monotonic if

$W(S;P) \leq W(T;\{Q-T : Q \in P\} \cup \{T\}), S \subseteq T$.

A game is w-partition monotonic if

$W(S;P) \leq W(S;Q)$ whenever Q is a refinement of P, i.e.,

$R \in Q \Rightarrow \exists S \in P$ such that $R \subseteq S$.

$(T;Q) > (S;P)$ if $S \subseteq T$ and $Q-\{T\}$ is a refinement of $P-\{S\}$.

$\{R-T : R \in P-\{S\}\}$

2

A unanimity game for (S;P) is one in which W(S;P)=1, W(T;Q)=1 if (T;Q) > (S;P) and all other ECL's=0.

Axioms and Definitions:

Let S and T be defined as subsets of N, P and Q as partitions of N, and W and V as games on N. The definitions are given as conditions which hold (for all games in class G). The reference to the class G in parentheses has been omitted from each definition.

Definition 1) Suppose $\pi: N \rightarrow N$ is any permutation of the set of players. Then $\pi$ acts as a permutation on CL and ECL in the following way:

$\pi(S) = \{\pi(j) \mid j \in S\}, \forall S \in CL$, and

$\pi(S^1, \{S^1, \ldots, S^k\}) = (\pi(S^1), \{\pi(S^1), \ldots, \pi(S^k)\})$,

$(S^1, \{S^1, \ldots, S^k\}) \in ECL$.

**Symmetry:** the payoff to a player will not change if the names of the players are permuted. Thus, $\forall j \in N$ and for every game W, $\Phi_j(W) = \Phi_{\pi(j)}(\pi O w)$ where $\pi O w$ is the game that results from relabelling W.

**Linearity:** $\Phi_i(W) = \Sigma_{(S;P) \in ECL} b(i,S;P)\ W(S;P)$. $\Phi(aW+bV) = a\Phi(W)+b\Phi(V)$

**Efficiency:** $\Phi_1(W) + \ldots + \Phi_n(W) = W(N;N)$.

Definition 2) Player j is a dummy player in game W if W(S;P)= W(S-{j};Q) for each $(S;P) \in ECL$ such that $j \in S$ and $|\delta| \geq 2$, where Q is a partition resulting from the removal of j to another, possibly empty, set of P.

**Dummy:** if player j is a dummy in game W, then $\Phi_j(W)=0$.

Definition 3) For any $P \in PT$ and $Q \in PT$, $P \wedge Q \in PT$ is defined as $P \wedge Q = \{S \cap T \mid S \in P, T \in Q, S \cap T \neq \emptyset\}$.

Definition 4) Given $W \in R^{ECL}$ and $S \in CL$, S is a carrier of W iff

$W(T;Q) = W(S \cap T;\ Q \wedge \{S, N/S\}),\ \forall (T;Q) \in ECL.$

**Carrier:** $\forall S \in ECL$, if S is a carrier of W, then

$\Sigma_{n \in S} \Phi_n(W) = W(N;N)$.

Definitions 1,3,4 and the carrier axiom are from Myerson (1977).

Definition 5) The dummy extension of W is the game $W^d$ defined on the player set $N \cup \{d\}$ by $W^d(S;P) = W(S-\{d\};\{R-\{d\}:R \in P\})\ \forall$ ECLS (S:P).

**Dummy Independence:** $\forall d \notin N$, and $\forall i \in N$, $\Phi_i(W^d) = \Phi_i(W)$.

Definition 5 and the dummy independence axiom are from Bolger (1987).

**Aggregate Monotonicity:** if $W(N;N) \geq V(N;N)$ and $W(S;P) = V(S;P)$, ~~then~~ $\forall (S;P) \neq (N;N)$, then $\Phi_i(W) \geq \Phi_i(V)\ \forall\ i \in N$.

**Group Monotonicity:** if $W(S;P) \geq V(S;P)$ and $W(T;Q) = V(T;Q)$, $\forall (T;Q) \neq (S;P)$, then $\Phi_i(W) \geq \Phi_i(V)$, $\forall i \in S$.

**Complementary Group Monotonicity:** if $W(S;P) \geq V(S;P)$ and $W(T;Q) = V(T;Q)$, $\forall (T;Q) \neq (S;P)$, then $\Phi_i(W) \leq \Phi_i(V)$, $\forall i \notin S$.

Definition 6) In a game W, the marginals for player i are the quantities $W(S;P) - W(S-\{i\};Q)$, where $i \in S$ and Q is a partition resulting from starting with partition P and moving i from S to another, possibly empty, set in P.

**Strong Monotonicity:** if each marginal for player i on game W is greater than or equal to the corresponding marginal for player i on game V, then $\Phi_i(W) \geq \Phi_i(V)$.

**Marginalist:** if the corresponding marginals for player i are the same on two games W and V, then the allocation $\Phi_i$ should be the same for both games, $\Phi_i(W) = \Phi_i(V)$.

**Strong Marginalist:** let $i \in N$. If for each partition Q, where

$T \in Q$, $i \in T$ and the summations over all partitions P of N that can be obtained from Q by moving i from T into another, possibly empty, set we have $\Sigma[W(T;Q)-W(T-\{i\};P)] = \Sigma[V(T;Q)-V(T-\{i\};P)]$, then $\Phi_i(W) = \Phi_i(V)$.

<u>Research</u>:

We will now look at the case N=3 in detail. The basis for a 3 player game can be represented by unanimity games in a matrix in the following manner:

| S;P | $W^1$ | $W^2$ | $W^3$ | $W^4$ | $W^5$ | $W^6$ | $W^7$ | $W^8$ | $W^9$ | $W^{10}$ |
|---|---|---|---|---|---|---|---|---|---|---|
| 123;123 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 12;12,3 | 0 | 1 | 0 | 0 | 1 | 1 | 0 | 1 | 1 | 0 |
| 13;13,2 | 0 | 0 | 1 | 0 | 1 | 0 | 1 | 1 | 0 | 1 |
| 23;23,1 | 0 | 0 | 0 | 1 | 0 | 1 | 1 | 0 | 1 | 1 |
| 1;1,2,3 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 1 | 0 | 0 |
| 2;1,2,3 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 1 | 0 |
| 3;1,2,3 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 1 |
| 1;23,1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 |
| 2;13,2 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 |
| 3;12,3 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 |
| | | | | | | | | | | |
| player 1: | 1/3 | 1/2 | 1/2 | 0 | 1-2z | z | z | 1 | 0 | 0 |
| player 2: | 1/3 | 1/2 | 0 | 1/2 | z | 1-2z | z | 0 | 1 | 0 |
| player 3: | 1/3 | 0 | 1/2 | 1/2 | z | z | 1-2z | 0 | 0 | 1 |

Because this is an upper triangular matrix and the dimension is equivalent to the number of unanimity games, the games are linearly independent. Thus, they form a basis for N=3.

The following explanations are given for the allocations to players 1, 2 and 3.

$W^1$: Each player is given 1/3 because by symmetry each one must be given an equal amount and by efficiency they must all add up to 1.

$W^2$: Player 3 receives nothing because he is a dummy player. Players 1 and 2 each receive 1/2 because by symmetry they each receive the same amount and by efficiency their payoffs must add up to 1.

$W^3$ and $W^4$: These are the same games as $W^2$ with the players permuted, therefore they should receive the same payoffs.

$W^5$: By symmetry, players 2 and 3 should receive the same amount. Thus, if each is given the payoff of z, by efficiency player 1 should be given the payoff 1-2z.

$W^6$ and $W^7$: These are the same games as $W^5$ with the players permuted, therefore they should receive the same payoffs.

$W^8$: Because players 2 and 3 are dummy players, they receive 0. Thus, by efficiency, player 1 receives the payoff of 1.

$W^9$ and $W^{10}$: These are the same games as $W^8$ with the players permuted, therefore they should receive the same payoffs. Since unanimity games form a basis for any arbitrary game, w can be written $w=\Sigma_{i=1to10}k_iw^i$. By linearity, $\Phi_i(W)=\Sigma_{i=1to10}k_i\Phi(w^i)$. The following is the set of equations:

$w(123) = k_1+k_2+k_3+k_4+k_5+k_6+k_7+k_8+k_9+k_{10}$

$w(12;3) = k_2+k_5+k_6+k_8+k_9$

$w(13;2) = k_3+k_5+k_7+k_8+k_{10}$

$w(23;1) = k_4+k_6+k_7+k_9+k_{10}$

$w(1;2,3) = k_5+k_8$

$w(2;1,3) = k_6+k_9$

$w(3;2,3) = k_7+k_{10}$

$w(1;23) = k_8$

$w(2;13) = k_9$

$w(3;12) = k_{10}$

From the allocations already given, $\Phi_1(W)= (1/3)k_1+ (1/2)k_2+ (1/2)k_3+ (1-2z)k_5+ (z)k_6+ (z)k_7+ k_8$. After solving for $k_1, \ldots ,k_{10}$ and substituting the answers into the above equation, the following result is found:

$$\Phi_1(W) = (1/3)[w(123)-w(23;1)] +(1/6-z)[w(12;3)-w(2;1,3)]+(z)[w(12;3)-w(2;13)] +(1/6-z)[w(13;2)-w(3;1,2)]+(z)[w(13;2)-w(3;12)] +(1/3-2z)[w(1;2,3)] + (2z)[w(1;23)]$$

The value of z is not defined if only the axioms of efficiency, linearity, symmetry and dummy are applied. Therefore, more axioms must be used in order to limit the value of z. Group, complementary group and strong monotone force $0 \leq z \leq 1/6$. Aggregate monotonicity places no restrictions on z as the payoff for player 1 in the unanimity game w(123) is always positive. Neither dummy independence nor marginalism place any restrictions on z.

The carrier axiom gives a value of z=1/3. A simple proof goes as follows: Let b=w(123)=w(1;23), a=(12;3)=w(13;2)=w(1;2,3), and 0=w(23;1)=w(2;13)=w(3;12)=w(2;1,3)=w(3;1,2). As seen, player 1 is the carrier, therefore he should receive the total worth of the game. Thus, $\Phi_1(W)=W(N;N)$ becomes $(2z-2/3)(b-a)=0$. Assuming that $a\neq b$, $z=1/3$. It is easily seen that the value gained from the carrier axiom is not group, complementary group or strong monontonic.

If we look at the case of w-superadditive and w-coalition monotonic games, however, the previous result does not occur. W-superadditivity would imply that $w(123)\geq w(12;3)+w(3;12)$ and, thus, $b\geq a$. W-coalition monontonicity would imply, however, that $w(12;3)\geq w(1;23)$ and, thus, $a\geq b$. Therefore, $a=b$ and no restrictions are placed on z!

Finally, applying strong marginalism causes $z=1/12$. Given the game $w(123)=w(23;1)=2$ and $w(13;2)=1$, $\Phi_1(W)=1/6$. Also, given the game $v(123)=v(23;1)=v(13;2)=v(3;1,2)=2$, $\Phi_1(V)=2z$. Thus, by strong marginalism, $\Phi_1(W)=\Phi_1(V)$ which forces $z=1/12$. This value is clearly group, complementary group and strong group monontonic.

We will now look at N=4 in detail. Because the unanimity games form a 37 x 37 matrix, only the 7 different cases will be presented.

<u>Case 1</u>: $w(1234)=1$.

By symmetry each player should receive the same amount and by efficiency they all receive 1/4.

<u>Case 2</u>: $w(1234)=w(123;4)=1$.

Because player 4 is a dummy, he receives 0. By symmetry players 1, 2, and 3 each receive the same amount and by efficiency they each receive 1/3.

<u>Case 3</u>: $w(1234)=w(123;4)=w(124;3)=w(12;3,4)=1$.

By symmetry players 3 and 4 and players 1 and 2 should each receive the same amount. Therefore, if players 3 and 4 are each given the payoff a, players 1 and 2 should each receive (1/2-a) by efficiency.

<u>Case 4</u>: $w(1234)=w(123;4)=w(124;3)=w(12;3,4)=w(12;34)=1$.

Because players 3 and 4 are both dummy players, they receive 0. By symmetry players 1 and 2 each receive the same amount and by efficiency they each receive 1/2.

<u>Case 5</u>: $w(1234)=w(123;4)=w(124;3)=w(134;2)=w(12;3,4)=w(13;2,4)$
$=w(14;2,3)=w(1;2,3,4)=1$.

By symmetry players 2, 3, and 4 should each receive the same amount. If they are each given the payoff b, then by efficiency player 1 should receive (1-3b).

<u>Case 6</u>: $w(1234)=w(123;4)=w(124;3)=w(134;2)=w(12;3,4)=w(13;2,4)$
$=w(14;2,3)=w(14;23)=w(1;2,3,4)=w(1;23,4)=1$.

Let player 4 receive the payoff c. By symmetry players 2 and 3 should receive the same amount. Thus, if they are each given the payoff d, by efficiency player 1 should receive (1-c-2d).

<u>Case 7</u>: $w(1234)=w(123;4)=w(124;3)=w(134;2)=w(12;3,4)=w(13;2,4)$
$=w(14;2,3)=w(12;34)=w(13;24)=w(14;23)=w(1;2,3,4)$
$=w(1;23,4)=w(1;24,3)=w(1;34,2)=w(1;234)=1$.

Because players 2, 3, and 4 are dummy players, they each receive 0. Therefore, by efficiency player 1 receives the payoff 1.

Using the same procedure as was followed in the case N=3 yields the following result for $\Phi_1(W)$:

$(1/4)\ [w(1234)]$

$+\ (1/12)\ [w(123;4) + w(124;3) + w(134;2)]$

$$- (1/4)\ [w(234;1)]$$

$$+ (1/12-a)\ [w(12;3,4) + w(13;2,4) + w(14;2,3)]$$

$$+ (a-1/12)\ [w(23;1,4) + w(24;1,3) + w(34;1,2)]$$

$$+ (a)\ [w(12;34) + w(13;24) + w(14;23)]$$

$$- (a)\ [w(23;14) + w(24;13) + w(34;12)]$$

$$+ (1/4+3a-3b)\ [w(1;2,3,4)]$$

$$- (1/12+a-b)\ [w(2;1,3,4) + w(3;1,2,4) + w(4;1,2,3)]$$

$$- (a-3b+c+2d)\ [w(1;23,4) + w(1;24,3) + w(1;34,2)]$$

$$+ (a-b+d)\ [w(2;13,4) + w(2;14,3) + w(3;12,4) + w(3;14,2) + w(4;12,3) + w(4;13;2)]$$

$$- (a+b-c)\ [w(2;34,1) + w(3;24,1) + w(4;23,1)]$$

$$- (6b-3c-6d)\ [w(1;234)]$$

$$+ (2b-c-2d)\ [w(2;134) + w(3;124) + w(4;123)].$$

If player 1 is a dummy, then one could set up the following game: $w(4;123)=w(4;1,23)=w(14;23)=1$. Player 1 should receive a payoff of 0 which results in the relationship $d=(b/2)$. Replacing d with $(b/2)$ and writing $\Phi_1(W)$ as the product of its marginals yields the following result:

$$\Phi_1(W)= (1/4)\ [w(1234) - w(234;1)]$$

$$+ (1/12-a)\ [w(123;4) - w(23;1,4)]$$

$$+ (a)\ [w(123;4) - w(23;14)]$$

$$+ (1/12-a)\ [w(124;3) - w(24;1,3)]$$

$$+ (a)\ [w(124;3) - w(24;13)]$$

$$+ (1/12-a)\ [w(134;2) - w(34;1,2)]$$

$$+ (a)\ [w(134;2) - w(34;12)]$$

$$+ (1/12+a-b)\ [w(12;3,4) - w(2;1,3,4)]$$

$$+ (-a+b/2)\ [w(12;3,4) - w(2;13,4)]$$

$$+ (-a+b/2)\ [w(12;3,4) - w(2;14,3)]$$

$$+ (1/12+a-b)\ [w(13;2,4) - w(3;1,2,4)]$$

$$+ (-a+b/2)\ [w(13;2,4) - w(3;12,4)]$$

$$+ (-a+b/2)\ [w(13;2,4) - w(3;14,2)]$$

$$+ (1/12+a-b)\ [w(14;2,3) - w(4;1,2,3)]$$

$$+ (-a+b/2)\ [w(14;2,3) - w(4;12,3)]$$

$$+ (-a+b/2)\ [w(14;2,3) - w(4;13;2)]$$

$$+ (a+b-c)\ [w(12;34) - w(2;34,1)]$$

$$+ (-b+c)\ [w(12;34) - w(2;134)]$$

$$+ (a+b-c)\ [w(13;24) - w(3;24,1)]$$

$$+ (-b+c)\ [w(13;24) - w(3;124)]$$

$$+ (a+b-c)\ [w(14;23) - w(4;23,1)]$$

$$+ (-b+c)\ [w(14;23) - w(4;123)]$$

$$+ (1/4+3a-3b)\ [w(1;2,3,4)]$$

$$+ (-a+2b-c)\ [w(1;23,4) + w(1;24,3) + w(1;34,2)]$$

$$+ (-3b+3c)\ [w(1;234)].$$

Group, complementary group and strong monotonicity place the following restrictions on the remaining three parameters:

$$0 \le a \le 1/12\ ;\quad 2a \le b \le a+1/12\ ;$$

$$b \le c \le a+b \text{ and } b \le c \le 2b-a.$$

Applying the dummy independence axiom yields the equation

$$\Phi_1(W) = (1/3)[w(123)] + (1/6)[w(12;3)+w(13;2)] - (1/3)[w(23;1)]$$

$$+ (1/3+b-2c)[w(1;2,3)] - (1/6+b/2-c)[w(2;1,3)+w(3;1,2)]$$

$$- (b-2c)[w(1;23)] + (b/2-c)[w(2;13)+w(3;12)].$$

Thus, $z=(-b/2+c)$.

The carrier axiom gives the values $a=1/4$, $b=1/6$ and $c=5/12$. The game $w(1234)=w(12;34)=1$ has players 1 and 2 as carrier.

dummy indiscernible



✓ $\frac{1}{4}\,[w(123) - w(23;1)]$

✓ $(\frac{1}{12} - a)\,[w(123) - w(23;1)$

✓ $a\,[w(123) - w(23;1)]$

✓ $(\frac{1}{12} - a)\,[w(12;3) - w(2;1,3)]$

✓ $a\,[w(12;3) - w(2;13)]$

✓ $(\frac{1}{12} - a)\,[w(13;2) - w(3;1,2)]$

✓ $(a)\,[w(13;2) - w(3;12)]$

✓ $(\frac{1}{12} + a - b)\,[w(12;3) - w(2;1,3)]$

✓ $(-a + b/2)\,[w(12;3) - w(2;13)]$

✓ $(-a + b/2)\,[w(12;3) - w(2;1,3)]$

✓ $(\frac{1}{12} + a - b)\,[w(13;2) - w(3;1,2)]$

✓ $(-a + b/2)\,[w(13;2) - w(3;12)]$

✓ $(-a + b/2)\,[w(13;2) - w(3;1,2)]$

✓ $(\frac{1}{12} + a - b)\,[w(1;2,3)$

✓ $(-a + b/2)\,[w(1;2,3)$

✓ $(-a + b/2)\,[w(1;2,3)$

✓ $(a + b - c)\,[w(12;3) - w(2;3,1)]$

✓ $(-b + c)\,[w(12;3) - w(2;13)]$

✓ $(a + b - c)\,[w(13;2) - w(3;2,1)]$

✓ $(-b + c)\,[w(13;2) - w(3;12)]$

✓ $(a + b - c)\,[w(1;23) -$

✓ $(-b + c)\,[w(1;23)$

✓ $(1/4 + 3a - 3b)\;w(1;2,3)$

✓ $(-a + 2b - c)\,[w(1;23) + w(1;2,3) + w(1;2,3)]$

✓ $(-3b + 3c)\;w(1;23)$

$[w(123) - w(23;1)]$

$(\frac{1}{4} + \frac{1}{12} - a + a) = \frac{1}{3} = \frac{1}{3}$

$[w(12;3) - w(2;1,3)]$

$\frac{1}{12} - a + \frac{1}{12} + a - b - a + \frac{b}{2} + a + b - c$

$\frac{1}{6} + \frac{1}{2}b - c = \frac{1}{6} - z$

$[w(13;2) - w(3;1,2)]$

$\frac{1}{12} - a + \frac{1}{12} + a - b - a + \frac{b}{2} + a + b - c$

$\frac{1}{6} + \frac{1}{2}b - c = \frac{1}{6} - z$

$[w(12;3) - w(2;13)]$

$a - a + \frac{b}{2} - b + c$

$-\frac{1}{2}b + c = z$

$[w(13;2) - w(3;12)]$

$a - a + \frac{b}{2} - b + c$

$-\frac{1}{2}b + c = z$

$[w(1;2,3)]$

$\frac{1}{12} + a - b - a + \frac{b}{2} - a + \frac{b}{2} + \frac{1}{4} + 3a - 3b$

$-2a + 4b - 2c$

$\frac{1}{3} + b - 2c = \frac{1}{3} - 2z$

$[w(1;23)]$

$a + b - c - b + c - a + 2b - c - 3b + 3c$

$-b + 2c = 2z$

---

$z = -\frac{1}{2}b + c$

$\alpha = -\frac{1}{2}(2\beta_1 + 2\beta_2) + 2\beta_1 + 2\beta_2 + \beta_3$

$= \beta_1 + \beta_2 + \beta_3$

Therefore, $0=\Phi_3(W)=(1/4-a)$ which implies that $a=1/4$. If player 1 is a carrier, then the following two games are possible: $w(12;3,4)=w(1;2,3,4)=w(13;2,4)=w(14;2,3)=1$ and $w(123;4)=w(1;23,4)=w(14;23)=1$. Because player 1 is the carrier, he receives the full worth of the game. Thus, the first game yields the equation $(1/2-3b)=0$ and the second $(1/12+2b-c)=0$. Therefore, $b=1/6$ and $c=5/12$.

Although this approach to partition function form games yields payoffs for all the players in the game, a few difficulties are seen. The two major weaknesses are the extremely large matrices introduced by games with many players and the fact that some variables drop out, as in the four player game when $d=b/2$. Thus, a new approach will now be considered in which the linearity and dummy axioms are used to imply marginalism. Then the symmetry and efficiency axioms are added to find a recursion relationship for the coefficients. This recursive formula still has parameters, but no variables are eliminated and the large matrices are no longer needed. For clarity, P is no longer all the partitions in the game, but P is now the partitions of N-S.

Linearity states that $\Phi_i(W)=\Sigma_{(S;P)\in ECL}\ b(i,S;P)W(S;P)$. Using this equation, the payoff to player one in the three player game would be written in the following manner:

$$\Phi_1(W)= b(1,123;\varnothing)W(123;\varnothing)+b(1,23;1)W(23;1)+ b(1,12;3)W(12;3)+b(1,2;1,3)W(2;1,3)+b(1,2;13)W(2;13)+ b(1,13;2)W(13;2)+b(1,3;1,2)W(3;1,2)+b(1,3;12)W(3;12)+ b(1,1;2,3)W(1;2,3)+b(1,1;23)W(1;23).$$

Combining linearity with the dummy axiom implies marginalism.

The proof goes as follows:

(Linearity) $\Phi_i(W)=\Sigma_{(S;P)\in ECL}\, b(i,S;P)W(S;P)$

$$=\Sigma_{(S;P)\in ECL(i)}\Big[ b(i,S;P)W(S;P)+ \Sigma_{R\in P\cup\{\varnothing\}} b(i,S-\{i\};P[i,R])W(S-\{i\};P[i,R]\Big])$$

where ECL(i) is all ECL's (S;P) s.t. $i\in S$ and P[i,R] is a partition identical to P except that i is added to $R \in P\cup\{\varnothing\}$.

$$=\Sigma_{(S;P)\in ECL(i)}\Big[ [b(i,S;P)+\Sigma_{R\in P\cup\{\varnothing\}} b(i,S-\{i\};P[i,R])]W(S;P)+ \Sigma_{R\in P\cup\{\varnothing\}} -b(i,S-\{i\};P[i,R])[W(S;P)-W(S-\{i\};P[i,R])]\Big]$$

Suppose W(N;N)=W(N-{i};{i})=1, then the payoff to player i as written in the form of the above formula would appear as:

$$\Phi_i(W)= [b(i,N;\varnothing)+b(i,N-\{i\};\{i\})]W(N;N)- b(i,N-\{i\},\{i\})\ W(N;N)-W(N-\{i\},\{i\})$$

$$= [b(i,N,\varnothing)+b(i,N-\{i\},\{i\})]$$

Since player i is a dummy player, $\Phi_i(W)=0$, thus $[b(i,N,\varnothing) + b(i,N-\{i\},\{i\})]=0$.

When unanimity games are viewed, they yield the same result, i.e. $\Phi_i(W)=\Sigma_{(S;P)\in ECL(i)}\ [b(i,S;P)+\Sigma_{R\in P\cup\{\varnothing\}} b(i,S-\{i\},P[i,R])]=0$ since i is a dummy player. Thus, if we define -b(i,S-{i};P[i,R]) =c(i,S;P;R), where R is the coalition in P to which i is added, the linearity and dummy axioms imply the following formula:

$$\Phi_i(W)=\Sigma_{(S;P)\in ECL(i)}\Sigma_{R\in P\cup\{\varnothing\}} c(i,S;P;R)[W(S;P)-W(S-\{i\};P[i,R])].$$

Thus, linearity and the dummy axiom imply marginalism. Also, due to the manner in which the games are set up they are w-superadditive, w-coalition and w-partition monotonic.

According to this formula the payoff to player one in a three player game would be written in the following manner:

$\Phi_1$(W)= c(1,123;∅;∅)[W(123;∅) -W(23;1)]

+ c(1,12;3;∅) [W(12;3) -W(2;1,3)]

+ c(1,12;3;3) [W(12;3) -W(2;13)]

+ c(1,13;2;∅) [W(13;2) -W(3;1,2)]

+ c(1,13;2;2) [W(13;2) -W(3;12)]

+ c(1,1;2,3;0) W(1;2,3) + c(1,1,23,0)W(1;23).

Applying symmetry further simplifies this formula. Let W(N;∅)=1 and all other ECL's=0. Thus, the payoff to player i is c(i,N;∅;∅). Now if the players are permuted, then the payoff to $\pi$(i) is c($\pi$(i),$\pi$(N);∅;∅). By symmetry, c(i,N;∅;∅) = c($\pi$(i),$\pi$(N);∅;∅). If the general game is set up where all ECL's which appear before W(S;P) when $\Phi_i$(W) is written as the sum of its marginal worths equal 1, W(S;P)=1 and all other worths = 0, including W(S-{i},P[i,R]), then the only place where player i is not a dummy is in W(S;P). Thus, the payoff to player i is c(i,S;P;R). Again, if the players are permuted, the payoff to $\pi$(i) is c($\pi$(i),$\pi$(S);$\pi$(P);$\pi$(R)). By symmetry, c(i,S;P;R) = c($\pi$(i),$\pi$(S);$\pi$(P);$\pi$(R)). Thus, the coefficient is not dependent on i. It is only dependent on the size of S, denoted by |S|, the sizes of the elements of P, denoted by ||P||, and the size of R, denoted by |R|. Hence, a new formula is found which can be stated:

$$\Phi_i(W)=\Sigma_{(S;P)\in ECL(i)}\Sigma_{R\in P\cup\{\emptyset\}}d(|S|;||P||;|R|)[W(S;P)-W(S-\{i\};P[i,R])]$$

where |R|=0 when S=i.

Writing the payoff for player one in a three player game according to this formula would appear as:

$\Phi_1$(W) = d(3;0;0) [W(123;∅) - W(23;1)]

$+ d(2;1;0)\ [W(12;3) - W(2;1,3)]$

$+ d(2;1;1)\ [W(12;3) - W(2;13)]$

$+ d(2;1;0)\ [W(13;2) - W(3;1,2)]$

$+ d(2;1;1)\ [W(13;2) - W(3;12)]$

$+ d(1;1,1;0)\ W(1;2,3) + d(1;2;0)\ W(1;23)$.

Finally, a recursion relation among the coefficients is found when efficiency is added. Let $W(N;\varnothing)=1$ and all other $W(S;P)=0$. All the players in the grand coalition look the same by symmetry and by efficiency their payoffs must add to 1, therefore $\Phi_1(W) = (1/n)$. According to efficiency, the sum of coefficients of $W(S;P)$ over all the players for coalitions other than grand one must be zero. The coefficient of player i for $W(N-\{i\},\{i\})$ however is $-(1/n)$, not zero. Thus the sum of the coefficients of all $j \in N$ s.t. $j \neq i$ must equal $(1/n)$. This leads to the equation:

$\Sigma_{r=0,1} d(n-1;1;r) = (1/n)$,

where $d(n-1;1;1)$ is the coefficient of $[W(N-\{i\};\{i\}) - W(N-\{i,j\};\{i,j\})]$ and $d(n-1;1;0)$ is the coefficient of $[W(N-\{i\},\{i\}) - W(N-\{i,j\};\{i\},\{j\})]$ for player j, by the previous formulas given by marginalism and symmetry.

Looking at the more general case yields a similar type of equation. As stated earlier, the sum of coefficients for all $W(S;P)$, other than the grand coalition, over all the players must be zero. This is easily seen when only linearity is applied. By linearity, $\Phi_i(W) = \Sigma_{(S;P)\in ECL}\, b(i,S;P)W(S;P)$. Thus, by efficiency, $\Sigma_{i\in N} b(i,S;P)=1$ if $S=N$ and 0 otherwise. When the payoff to player i is written as the sum of its marginal worths, the coefficients for the different players $j \in S$, given a particular P, appear in various places in the payoff to player i. When the general formula given by symmetry and marginalism is viewed, the coefficient on any W(S;P) for player i is $\Sigma_{R\in P\cup\{\emptyset\}} d(|S|;||P||;|R|)$ and not zero. By efficiency, the sum over all players $j\in N$ s.t. $j\in T$ where $|T| = |S|$ and $||N-T|| = ||P||$ must equal $\Sigma_{R\in P\cup\{\emptyset\}} d(|S|;||P||;|R|)$. This leads to the relationship:

$$|S|\Sigma_{R\in P\cup\{\emptyset\}} d(|S|;||P||;|R|) = \Sigma_{R\in P\cup\{\emptyset\}} |R| d(|S+1|;||P-\{R\}\cup\{R-1\}||;|R-1|)$$

where $d(|S|;||P||;0)=\Sigma_{R\in P}[(1/|S|)d(|S+1|;||P-\{R\}\cup\{R-1\}||;|R-1|)-d(|S|;||P||;|R|)]$.

There are |S| ways to pick a player $j\in S$, therefore it is multiplied by |S|. The same applies to the coefficient |R|. The size of a coalition and particular partitions will have occurred before when by writing the payoff to player i by its marginal worths, {i} was taken out of a coalition T, where $|T|=|S+1|$ and $T\in ECL(i)$, and placed in a coalition Q, where $|Q|=|R-1|$. This leads to the relationships between |S| and |S+1|, ||P|| and $||P-\{R\}\cup\{R-1\}||$, and |R| and |R-1|.

As stated earlier, d(n,0,0)=(1/n) by efficiency and symmetry. Thus, a recursive relationship is found whereby the coefficients are determined by previous ones and the number of parameters introduced is all $(x,y \in ||P||) - 1$ s.t. $x\neq y$.

According to the final formula, the payoff to player 1 in a three player game is:

| Equation: | | $\Phi_1(W)=$ |
|---|---|---|
| d(3;0;0)=(1/3) | ⇒ | $(1/3)W(123,\emptyset)+$ |
| 2[d(2;1;0)+d(2;1;1)]=d(3;0;0) | ⇒ | $(1/6 - \mu_1)[W(12;3)-W(2;1,3)]+$ |

$\mu_1\ [W(12;3)-W(2;13)]+$

$(1/6-\mu_1)[W(13;2)-W(3;1,2)]+$

$\mu_1\ [W(13;2)-W(3;12)]+$

$d(1;1,1;0)=2d(2;1;0) \quad \Rightarrow \quad (1/3-2\mu_1)\ W(1;2,3)+$

$d(1;2;0)=2d(2;1;1) \quad \Rightarrow \quad 2\mu_1\ W(1;23).$

Using the recursion relation, the payoff to player 1 in a four player game would appear as follows:

Equation: $\quad \Phi_1(W)=$

$d(4;0;0)=(1/4) \quad \Rightarrow \quad (1/4)W(1234;\varnothing)+$

$3[d(3;1;0)+d(3;1;1)]=d(4;0;0) \quad \Rightarrow \quad (1/12-\mu_1)[W(123;4)-W(23;1,4)]+$

$\mu_1\ [W(123;4)-W(23;14)]+$

$(1/12-\mu_1)[W(124;3)-W(24;1,3)]+$

$\mu_1\ [W(124;3)-W(24;13)]+$

$(1/12-\mu_1)[W(134;2)-W(34;1,2)]+$

$\mu_1\ [W(134;2)-W(34;12)]+$

$2[d(2;2;2)+d(2;2;0)]=2d(3;1;1) \quad \Rightarrow \quad (\mu_1-\mu_2)[W(12;34)-W(2;34,1)]+$

$\mu_2\ [W(12;34)-W(2;134)]+$

$(\mu_1-\mu_2)[W(13;24)-W(3;24,1)]+$

$\mu_2\ [W(13;24)-W(3;124)]+$

$(\mu_1-\mu_2)[W(14;23)-W(4;23,1)]+$

$\mu_2\ [W(14;23)-W(4;123)]+$

$2[2d(2;1,1;1)+d(2;1,1;0)]=2d(3;1;0) \quad \Rightarrow$

$(1/12-\mu_1-2\mu_3)[W(12;3,4)-W(2;1,3,4)]+$

$\mu_3\ [W(12;3,4)-W(2;13,4)]+$

$\mu_3\ [W(12;3,4)-W(2;14,3)]+$

$(1/12-\mu_1-2\mu_3)[W(13;2,4)-W(3;1,2,4)]+$

$\mu_3\ [W(13;2,4)-W(3;12,4)]+$

$$\mu_3\ [W(13;2,4)-W(3;14,2)]+$$

$$(1/12-\mu_1-2\mu_3)[W(14;2,3)-W(4;1,2,3)]+$$

$$\mu_3\ [W(14;2,3)-W(4;12,3)]+$$

$$\mu_3\ [W(14;2,3)-W(4;13,2)]+$$

$$d(1;3;0)=3d(2;2;2) \quad \Rightarrow \quad 3\mu_2\ [W(1;234)]+$$

$$d(1;2,1;0)=d(2;2;0)+2d(2;1,1;1) \Rightarrow (\mu_1-\mu_2+2\mu_3)[W(1;23,4)+W(1;24,3)$$

$$+W(1;34,2)]+$$

$$d(1;1,1,1;0)=3d(2;1,1;0) \quad \Rightarrow (1/4-3\mu_1-6\mu_3)[W(1;2,3,4)].$$

Conclusion:

The last method was briefly introduced without any other axioms, such as dummy independence, carrier, and strong marginalism, being applied. It would be very desirable to have a simple result when any of these axioms were applied. Further research could be done in this area by applying these axioms or finding a simpler formula that did not depend on recursion.

Not only are there specific questions unanswered in respect to this formula, but there are also many general questions in the field of partition function form games. It would be simpler if an axiom was found which would yield a payoff vector without parameters. This would be ideal in the sense of a shapley extension on partition function form games, the major theme of the paper.

*18

**References**

Bolger, E.M.: Values for Games in Partition Function Form. Society for Industrial and Applied Mathematics, 1987,8:460-466

Bolger, E.M.: A Set of Axioms for a Value for Partition Function Games. International Journal of Game Theory, 1989,18:37-44

Lucas, W.F., and R.M. Thrall: n-Person Games in Partition Function Form. Naval Research Logistics Quarterly X, 1963,281-298.

McCaulley, P.: Axioms and Values on Partition Function Form Games. REU Cooperative Game Theory, 1990.

Myerson, R.B.: Values of Games in Partition Function Form. International Journal of Game Theory, 1977,6:23-31.