The Machiavelli Index

1. Introduction

"Upon this, one has to remark that men ought either to be well treated or crushed, because they can avenge themselves of lighter injuries, of more serious ones they cannot; therefore the injury that is to be done to a man ought to be of such a kind that one does not stand in fear of revenge."

-Niccolo Machiavelli, The Prince

The Machiavelli power index is a way of measuring relative *a priori* voting power in a decision-making assembly. The key idea behind this index is that groups of voters can sometimes increase their power by forming a bloc: an agreement to decide amongst themselves how all will vote in the full assembly. Voters seeking to maximize power will seek to form blocs, and we will measure an individual's voting power by an average of the power obtainable from blocs that are likely to form.

The Machiavelli power index has a different viewpoint from power indices previously proposed in the literature. The Shapley-Shubik (1954) power index imagines voters joining a coalition one by one rather than together forming a bloc. The voter who changes the coalition from losing to winning (said to be pivotal) has all of the voting power for a particular order of players. The Shapley-Shubik voting power for a particular player is the number of times that player is pivotal divided by all possible orders of the players.

Three power indices in the literature can be interpreted as assuming that certain blocs can form, pivotal voters in those blocs (those whose unilateral defection from the bloc changes it from winning to losing) share all power equally, and the power index is calculated by averaging the power obtained in each bloc that can form. The Johnston (1978) power index assumes that each winning coalition is equally likely to form. The Deegan-Packel (1978) power index assumes that each minimal winning coalition is equally likely to form. The Banzhaf (1965) power index assumes that the probability that a winning coalition will form is proportional to the number of pivotal voters in the winning coalition. The Machiavelli power index differs both (1) in what blocs are assumed to form, and (2) in that the power of a voter in a bloc depends on its structure and is obtained recursively rather than assumed to be equally shared with others.

In section 2, we review the definitions of power indices previously reported in the literature. In section 3, we provide an intuitive and formal definition of the Machiavelli power index. In section 4, we determine the Machiavelli power index for all proper simple games with four or fewer voters. In section 5, we illustrate some difficulties with our definition of the Machiavelli power index. In section 6, we offer ideas for future research.

A Motivating Real World Example

The United Nations Security Council is an example of a real world application of power indices. The United Nations Security Council was formed shortly after World War II to prevent wars of that size from ever happening again.There are five permanent members on the Council: The United States, Great Britain, Russia, China, and France. Ten non-permanent members have two year terms on the Council. To pass some bill, consent is required from all five permanent members and four of the ten non-permanent members.

General Formal Definitions

A ***proper simple game*** is a set of players, N, and a set of winning coalitions, W, that satisfies:

1. Winning coalitions are sets of players (if $S \in W$, then $S \subseteq N$),

2. Unanimity wins ($N \in W$),

3. No player is detrimental (if $A \subseteq B$ and $A \in W$, then $B \in W$), and

4. Only one coalition can be winning at any particular time (if $A \in W$, then the complement $N \backslash A \notin W$).

A ***minimal winning coalition*** is a set of players, M such that:

1. The coalition is winning, ($M \in W$),

2. If any player is removed, the coalition is no longer winning, (for every player $j \in M$, $M - \{ j \} \notin W$).

$\boldsymbol{M_A}$ is the set of all M, (all $M \in M_A$).

Then, a proper simple game (N, W) can instead be written as $\mathcal{W}^{\min} = \{M_1, M_2, \ ..., M_n\}^+$ where M_k is some group of players k that is a minimal winning coalition.

A ***dummy*** is a player i, where:

1. i is in no minimal winning coalitions, ($i \notin M$ for all $M \in W$).

A ***unanimity game*** is a proper simple game where:

1. All players' approval is required to pass some bill, (for every player $j \in N$, $N - \{ j \} \notin W$).

A ***weighted voting game*** is a proper simple game (N, W) such that

1. $W = [q; w_1, w_2, ..., w_n]$, where q is the number of votes needed to win, and w_i is the number of votes possessed by some player i.

2. Power Indices in the Literature

In this section, we will use the example $\mathcal{W}^{\min} = \{AB, \ AC, \ BCD\}^+$. This could also be written as [5; 3, 2, 2, 1] to illustrate four power indices in the literature.

The Shapley Shubik Index

The Shapley Shubik Index, or the S-S index, calculates power by assuming that voters in some assembly vote in order. The nth voter who changes a losing coalition to a winning coalition is termed pivotal. The S-S index assumes that all orderings of voters are equally likely. A players' power is the number of times that that player is pivotal divided by the total number of permutations.

More formally, the S-S index is defined as follows. For some proper simple game (N, W), let π be some permutation of the group of voters N={1, 2, 3, ..., n}. Then {π(1), π(2),π(3), ..., π(n)} is some reordering of N. It follows that there must be a unique k such that π(k) is pivotal in the coalition {π(1), π(2),π(3), ..., π(k)}. With k, the coalition is winning, without k, the coalition is losing. We call this unique π(k) pivotal for this particular permutation, p(π). Then the S-S index for some voter i in the proper simple game W, denoted $\rho_i(W)$, is equal to the number of times i is pivotal in all permutations, divided by the total number of permutations. In other words,

$$\rho_i(W) = \frac{|\{\pi : p(\pi) = i\}|}{n!}$$

Example: $\mathcal{W}^{\min} = \{AB,\ AC,\ BCD\}^{+}$

In a four player voting game, there are 24 possible orders that the voters could vote in, 4!=24. All 24 orders are listed below, with the pivotal player bolded.

A**B**CD
A**B**DC
A**C**BD
A**C**DB
AD**B**C
AD**C**B
B**A**CD
B**A**DC
BC**A**D
BC**D**A
BD**A**C
BD**C**A
C**A**BD
C**A**DB
CB**A**D
CB**D**A
CD**A**B
CD**B**A
DA**B**C
DA**C**B
DB**A**C
DB**C**A
DC**A**B
DC**B**A

A is pivotal 10 times, so $\rho_A(AB, AC, BCD)^+ = \frac{10}{24} = 41.67\%$

B is pivotal 6 times, so $\rho_B(AB, AC, BCD)^+ = \frac{6}{24} = 25\%$

C is pivotal 6 times, so $\rho_C(AB, AC, BCD)^+ = \frac{6}{24} = 25\%$

D is pivotal 2 times, so $\rho_D(AB, AC, BCD)^+ = \frac{2}{24} = 8.33\%$

The Relative Banzhaf Index

Unlike the S-S index, the relative Banzhaf index, or Bz index, is not concerned with the order in which players may vote. First the Bz index calculates the number of times that a given player is pivotal in all combinations of voters. Then, that is divided by the sum of the number of times pivotal for all players.

Formally, we first consider some proper simple game (*N*, *W*). The Bz score, η_i for some player *i* is the number of times that *i* is pivotal.

$\eta_i\ (N,\ W) = |\ \{S \subseteq N\text{: } S \in W \text{ and } S - \{\ i\ \} \notin W\ \}\ |$

The relative Bz index for some voter *i* in the proper simple game (*N*, *W*) is

$\beta_i\ (N,\ W) = \frac{\eta_i(N,\ W)}{\sum_{j=1}^{n} \eta_j(N,\ W)}$

Example: $\mathcal{W}^{\min} = \{AB,\ AC,\ BCD\}^+$

Below, all possible combinations of {A, B, C, D} are listed. In each coalition, players are bolded, who, if taken away would change the coalition from winning to losing.

A
B
C
D
AB
AC
AD
BC
BD
CD
ABC
ABD
ACD
BCD
ABCD

A is pivotal 5 times, so β_A (AB, AC, BCD)$^+$ = $\frac{5}{12}$ = 41.67%

B is pivotal 3 times, so β_B (AB, AC, BCD)$^+$ = $\frac{3}{12}$ = 25%

C is pivotal 3 times, so β_C (AB, AC, BCD)$^+$ = $\frac{3}{12}$ = 25%

D is pivotal 1 times, so β_D (AB, AC, BCD)$^+$ = $\frac{1}{12}$ = 8.33%

Note that the S-S index and the Bz index will not always yield the same answer, even though they do in this particular example.

The Deegan and Packel Index

The Deegan and Packel Index, or the D-P index, assumes that only minimal winning coalitions will form, that all minimal winning coalitions have an equal probability of forming, and that the members of a minimal winning coalition benefit equally from winning. Each player gets $\frac{1}{n}$ points added to its score for every minimal winning coalition, where n is the number of players in that minimal winning coalition. Then, that score is divided by the total number of minimal winning coalitions.

Formally, let M_{A_i}be the set of minimal winning coalitions that some player *i* belongs to. So $M_{A_i} = \{S \in M_A : i \in S\}$. In addition, let the number of players in some minimal winning coalition *L* be written as $|M_L|$. The D-P index δ_i (*N, W*) for voter *i* in some proper simple game (*N, W*) is:

$\delta_i\ (N,\ W) = \frac{1}{M_A} \times \sum_{L=1}^{|M_A|} \frac{1}{|M_L|}$.

Example: $\mathcal{W}^{\min} = \{AB,\ AC,\ BCD\}^+$

There are three minimal winning coalitions in this game. They are listed below.

AB
AC
BCD

A gets $\frac{1}{2}$in both the AB and AC minimal winning coalitions, for a total sum of 1. This is then divided by three, the number of minimal winning coalitions. δ_A (AB, AC, BCD)$^+$ = $\frac{1}{3} \times \sum_{L=1}^{3} \frac{1}{|M_L|}$ = $\frac{1}{3}$ = 33.33%.
B gets $\frac{1}{2}$ in the AB coalition and $\frac{1}{3}$ in the BCD coalition for a total of $\frac{5}{6}$. This is then divided by three, the number of minimal winning coalitions. δ_B(AB, AC, BCD)$^+$ = $\frac{1}{3} \times \sum_{L=1}^{3} \frac{1}{|M_L|}$ = $\frac{5}{18}$ = 27.78%.
C gets $\frac{1}{2}$ in the AC coalition and $\frac{1}{3}$ in the BCD coalition for a total of $\frac{5}{6}$. This is then divided by three, the number of minimal winning coalitions. δ_C(AB, AC, BCD)$^+$ = $\frac{1}{3} \times \sum_{L=1}^{3} \frac{1}{|M_L|}$ = $\frac{5}{18}$ = 27.78%.
C gets $\frac{1}{3}$ in the BCD coalition for a total of $\frac{1}{3}$. This is then divided by three, the number of minimal winning coalitions. δ_D (AB, AC, BCD)$^+$ = $\frac{1}{3} \times \sum_{L=1}^{3} \frac{1}{|M_L|}$ = $\frac{1}{9}$ = 11.11%.

The Johnston Index

The Johnston Index, or the Jn index, borrows ideas from both the Bz index and the D-P index. The only difference between the Jn index and the Bz index is that the Jn index adds $\frac{1}{n}$ to a given players score, where n is the number of pivotal players in that winning coalition.

Let p(*S*) be the number of pivotal players in some coalition *S*. Let S_q be some coalition q. The Jn score for voter *i* is $\zeta_i(N,\ W) = \sum_{q=1}^{|q|} \left\{ \frac{1}{p(S)} :\ S \in W \text{and} S - \{i\} \notin W \right\}$. The Jn index $\gamma_i(N,\ W) = \frac{\zeta_i(N,\ W)}{\sum_{j=1}^{n} \zeta_j(N,\ W)}$.

Example: $\mathcal{W}^{\min} = \{AB,\ AC,\ BCD\}^+$

Here are all possible coalitions. Pivotal players are bolded.

A
B
C

D
AB
AC
AD
BC
BD
CD
ABC
ABD
ACD
BCD
ABCD

A gets 1 in the ABC coalition, $\frac{1}{2}$in the AB, AC, ABD, and ACD coalitions, for a total sum of 3. This is then divided by six, the number of coalitions with pivotal players. $\gamma_A(\text{AB, AC, BCD})^+ = \frac{3}{6} = \frac{1}{2} = 50\%$. B gets $\frac{1}{2}$in the AB and ABD coalitions and $\frac{1}{3}$ in the BCD coalitions for a total sum of $\frac{4}{3}$. This is then divided by six, the number of coalitions with pivotal players. $\gamma_B(\text{AB, AC, BCD})^+ = \frac{4/3}{6} = \frac{4}{18} = \frac{2}{9} = 22.22\%$. C gets $\frac{1}{2}$in the AC and ACD coalitions and $\frac{1}{3}$ in the BCD coalitions for a total sum of $\frac{4}{3}$. This is then divided by six, the number of coalitions with pivotal players. $\gamma_C(\text{AB, AC, BCD})^+ = \frac{4/3}{6} = \frac{4}{18} = \frac{2}{9} = 22.22\%$. D gets $\frac{1}{3}$ in the BCD coalitions for a total sum of $\frac{1}{3}$. This is then divided by six, the number of coalitions with pivotal players. $\gamma_D(\text{AB, AC, BCD})^+ = \frac{1/3}{6} = \frac{1}{18} = 5.56\%$.

3. The Machiavelli Index

The set of ***veto power players*** is a group of players V such that

1. V_i is in all winning coalitions, $V_i \in W$ for all W.

A ***blocking coalition*** is a set of players B such that:

1. The complement of B is not winning, ($N \backslash B \notin W$),

2. B is not winning, ($B \notin W$),

3. Veto power players are not a part of the coalition, ($V_i \notin B$ for all $i \in V$).

A ***blocking bloc that blocks other blocking coalitions*** is a blocking coalition B_b such that

1. The set of players C outside of B_b and V are not in all winning coalitions, ($i \notin B_b \cup V$ implies $i \in C$, there exists an M such that for all $i \in C$ $i \notin M$).

2. Blocking blocs are assumed to form a unanimity game, to give all members veto power. (B_b implies $\mathcal{W}^{\min} = \{B_b \cup V\}^+$).

A ***bloc structure*** for some proper simple game (N, W) is a proper simple game (P, H) that causes a new proper simple game (N, W_H) where

1. The group of players P is a subset of the group of players N, ($P \subset N$),

2. R is some coalition that wins within the bloc, Q is some subset of the players outside the bloc, and when R wins, that causes P to win, and P and Q together are winning, (W_V={ $R \cup Q$: $R \in H$, $Q \subset N \backslash M$, and $P \cup Q \in W$ }).

The Machiavelli Index is a way of measuring *a priori* power in proper simple games. The key idea behind this index is that in a non-unanimity voting situation, some group of players can increase their power by forming a bloc. In unanimity games, blocs are pointless, because everyone has veto power.

The ***Machiavelli Power Index*, μ,** is defined recursively as follows, interspersed with comments and intuition. Let μ be defined for all proper simple games with fewer than n players.

In the case of an n-player unanimity game μ assigns $\frac{1}{n}$ power to all players $i \in N$ in the game, ($N \in M$ implies $\mu_i(N, \{M_u\}^+) = 1/n$ where there are n players i in the only minimal winning coalition M_u.

In games where there are more than one minimal winning coalition, reasonable bloc structures are compared to determine what players would actually do based upon their options and the options of their opponents. Players are assumed to be power-maximizing. Blocs can be classified as one of five types: minimal winning coalitions, non-minimal winning coalitions, blocking coalitions that do not allow other blocking coalitions, blocking coalitions that do allow other blocking coalitions, and blocs that are neither winning or blocking. Since players V in all winning coalitions already have veto power, they have nothing to gain by joining blocking blocs.

Non-minimal winning coalitions include players that aren't necessary to win. Intuitively, by removing these players, everyone else could benefit, or at least not lose power. Blocking blocs that allow other blocking blocs are fragile. For example in the game ABC, ADE, B and D could form a blocking bloc. This would cause minimal winning coalitions ABCD, ABDE. But, then C and E might form a blocking bloc too. Then it is a five player unanimity game ABCDE. It would be smarter for B and D to include C or E. If they include C, for example, then ABCD is the only minimal winning coalition. Then, B and D get $\frac{1}{4}$ of the power instead of $\frac{1}{5}$. Players that form a bloc that is neither winning nor blocking risk that the other players will form a winning bloc, leaving players in the impotent bloc with no power. These intuitive eliminations leave us with two kinds of coalitions to consider, minimal winning coalitions and blocking coalitions that prevent other blockers. Incidentally, blocking blocs that prevent other blockers imply that those blockers win with the set of all veto power players.

For non unanimity games, μ considers the set of blocs *Z,* where $Z = M_A \cup B_b$.

Another intuition is that some player i would not join in a bloc with some other player j if j gets more power in the bloc but had the same or less power than player i in the original game. The challenge then becomes figuring out the power order of the players in the original game. If two players are in all the same winning coalitions, they clearly have the same amount of *a priori* power. Similarly, if some player i is in all the winning coalitions that j is in and more, it is clear that i has at least as much power as j. Additionally, no player would join a bloc where they get no power.

A ***power consistent ordering*** for some proper simple game (N, W) is defined as follows.

Suppose $(N, \mathcal{W})$ is a proper simple game and π is a permutation of N. For each coalition $S \subset N$, define $\pi(S) = \{\pi(k) : k \in S\}$. Also define $\pi(\mathcal{W}) = \{\pi(S) : S \subset N\}$. We will call π an *automorphism* if $\pi(\mathcal{W}) = \mathcal{W}$.

Suppose $(N, \mathcal{W})$ is a proper simple game. Players i and j are ***symmetric*** if there is an automorphism π of N satisfying $\pi(i) = j$. The ***player power relation*** $\leq_{\mathcal{W}}$ is defined by $i \leq_{\mathcal{W}} j$ if for some automorphism π it follows that $S \cup \{\pi(i)\} \in \mathcal{W}$ implies $S \cup \{j\} \in \mathcal{W}$ for all $S \subset N \backslash \{\pi(i), j\}$.

The ***player power relation*** $\leq_{\mathcal{W}}$ is defined by $i \leq_{\mathcal{W}} j$ if there are players $i', j' \in N$ that are symmetric to i and j, respectively, and $S \cup \{i'\} \in \mathcal{W}$ implies $S \cup \{j'\} \in \mathcal{W}$ for all $S \subset N \backslash \{i', j'\}$.

In sections 4 and 5, the player power relation will be expressed in this notation: $\mu_i \geq \mu_j = \mu_k$ for some players *i*, *j*, and *k.*

Dummy players are assigned zero power.

Let $(N, \mathcal{W})$ be a proper simple game with *n* players for which there is more than one minimal winning coalition. If $(N, \mathcal{W})$ has at least one dummy player, then let *G* be the set of players that are not dummies, let $\mathcal{W}_G = \{S \subset G : S \in \mathcal{W}\}$, and define $\mu_i(N, \mathcal{W}) = \mu_i(G, \mathcal{W}_G)$ for all $i \in G$ and $\mu_i(N, \mathcal{W}) = 0$ for all $i \in N \backslash G$.

Upon determining important blocs that might form, we assign each bloc all reasonable, power ordered divisions of power that were found recursively.

For all minimal winning blocs *M*, assign all *M* all possible dummy-free, power consistent bloc structures. Call the set of these structures M_S.

For non unanimity games, μ considers the set of Z_S, where $Z_S = M_S \cup B_b$.

Some bloc *F dominates* some other bloc *H* if and only if every player in *F* has more power than they did in *H*, ($F = (P, Q)$, $H = (X, Y)$ for all $i \in P$, $\mu_i(N, \{W_Q\}^+) > \mu_i\ (N,\ \{W_Y\}^+$ implies $H \to F$).

For two bloc structures *A* and *B*, the notation $A ⥮ B$ indicates that both A→B and B→A are false.

Then a directed graph can be drawn with bloc structures as vertices and dominations as edges pointing from a dominated structure to a dominating structure. Then, a Von Neumann-Morgenstern idea is used to determine which structures are stable.

The ***stable set*** consists of a group of structures *S* such that

1. *S* is a subset of the group of blocs being considered Z_S, ($S \subseteq Z_S$),

2. Every structure *U* outside the stable set is directly dominated by some structure *T* inside the set, (for all *U*, there exists a *T* such that $U \to T$),

3. No structure *T* inside the stable set is dominated by another structure *R* inside the stable set, ($T ⥮ R$ for all *T* and *R* in *S)*.

An ***unstable structure*** *U* refers to a structure not in the stable set, $U \notin S$.

A ***stable structure*** S_1 refers to a structure in the stable set, $S_1 \in S$.

In practice, this means that undominated structures are stable, because all unstable structures must be dominated by a stable structure.

To see why these mapping rules make sense, consider this. Suppose that some structure K dominates another structure L. Assume, to the contrary, that both are stable. Because K dominates L, every player in the bloc K has more power than they do in the bloc L. If K is stable, all players i in the bloc K would benefit by moving to K, because each player i doesn't have to worry about moving to some other bloc Q where i gets less power than i did in L. Therefore, all players i in K cannot lose by moving from L to K. So, L and K cannot both be stable.

Suppose there is some structure X dominated only by some structure Y. Suppose Y is dominated only by some undominated structure Z. Z is clearly stable, because it is undominated. Assume, to the contrary, that X and Y are both unstable. There must be at least one player j in both X and Y because this is a proper simple game. If all players j that are in X and Y are also in Z, then Z must dominate X**.** If one or more of the players j in X and Y is not in Z, j would not be willing to move to bloc structure Y, because then other players would move to Z, leaving j with no power.

Finally, define $\mu(N, \mathcal{W})$ to be the average of $\mu(N, \mathcal{W}_H)$ over all stable bloc structures $(P, H) \in Z_S$.

Like other power indices, the Machiavelli Index is normalized so that all powers add up to 1.

4. The Index Illustrated Through Four Players

In this section, we will examine the fourteen proper simple games that have no dummies and no more than four players (Shapley, 1954). The games are labeled in the same way as they are in Shapley's paper (i.e. the game b. corresponds to the same game as in Shapley's paper). For shorthand, $\mu_A = \frac{1}{3}$ really means $\mu_A(\{A, B, C\}, \{\{A, B\}, \{A, C\}, \{B, C\}, \{A, B, C\}\}) = \frac{1}{3}$ in the case of game f.

b. $\mathcal{W}^{\min} = \{A\}^+$ implies $\mu(A, M) = (1)$

In a voting situation comprising just one player, there should be no debate as to how to attribute power. It is a dictator game.

c. $\mathcal{W}^{\min} = \{AB\}^+$ implies $\mu(AB, M) = \left(\frac{1}{2}, \frac{1}{2}\right)$

The only proper two player game is a unanimity game. Therefore $\mu_A = \frac{1}{2}$, $\mu_B = \frac{1}{2}$.

d. $\mathcal{W}^{\min} = \{ABC\}^+$ implies $\mu(ABC, M) = \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$

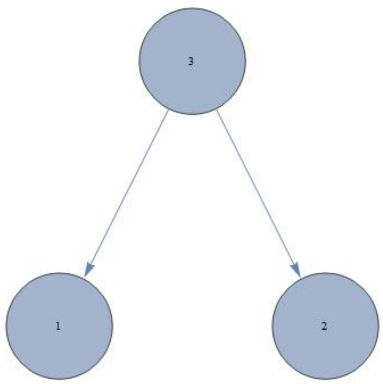
Again, we have a unanimity game and consequently its powers are $\mu_A = \frac{1}{3}$, $\mu_B = \frac{1}{3}$, $\mu_C = \frac{1}{3}$.

e. $\mathcal{W}^{\min} = \{AB, AC\}^+$ implies $\mu(ABC, M) = \left(\frac{1}{2}, \frac{1}{4}, \frac{1}{4}\right)$

We have arrived at Shapley's first non-unanimity game. For this game, minimal winning coalitions are AB and AC. Three blocs could form. They are in the chart below.

The power ordering for this game : $\mu_A \geq \mu_B = \mu_C$.

Index	Bloc	Type	Bloc MWCs	New full MWCs	Powers	Dominates	Dominated By
1	AB	MinWin	AB	AB	$\frac{1}{2}, \frac{1}{2}, 0$	3	-
2	AC	MinWin	AC	AC	$\frac{1}{2}, 0, \frac{1}{2}$	3	-
3	BC	Blocking	BC	ABC	$\frac{1}{3}, \frac{1}{3}, \frac{1}{3}$	-	1, 2



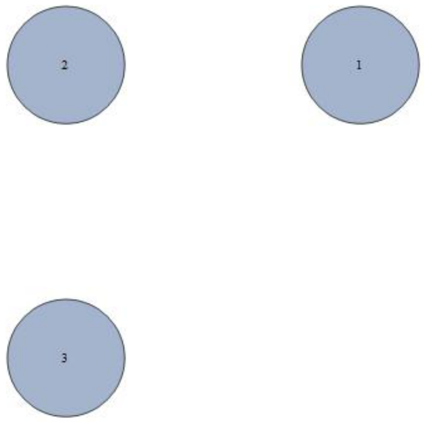
Here, if the BC bloc is agreed upon, A could lure either B or C away with the promise of $\frac{1}{2}$ the power instead of $\frac{1}{3}$. A would also benefit by moving away from structure 3 and to either 1 or 2. Therefore, structure 3 is unstable and should not be included in the average. A is indifferent between 1 and 2 so neither structure dominates the other. Thus, structures 1 and 2 are averaged to yield the answer $\mu_A = \frac{1}{2}$, $\mu_B = \frac{1}{4}$, $\mu_C = \frac{1}{4}$.

f. $\mathcal{W}^{\min} = \{AB, AC, BC\}^{+}$ implies $\mu(ABC, M) = \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$

This is the three player majority game with minimal winning coalitions AB, AC, BC.

$\mu_A = \mu_B = \mu_C$.

Index	Bloc	Type	Bloc MWCs	New full MWCs	Powers	Dominates	Dominated By
1	AB	MinWin	AB	AB	$\frac{1}{2}, \frac{1}{2}, 0$	-	-
2	AC	MinWin	AC	AC	$\frac{1}{2}, 0, \frac{1}{2}$	-	-
3	BC	MinWin	BC	BC	$0, \frac{1}{2}, \frac{1}{2}$	-	-



Since there are only undominated structures, all three bloc structures are averaged together. A three player majority game is assigned powers $\mu_A = \frac{1}{3}$, $\mu_B = \frac{1}{3}$, $\mu_C = \frac{1}{3}$.

g. $\mathcal{W}^{\min} = \{ABCD\}^{+}$ implies $\mu(ABCD, M) = \left(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}\right)$

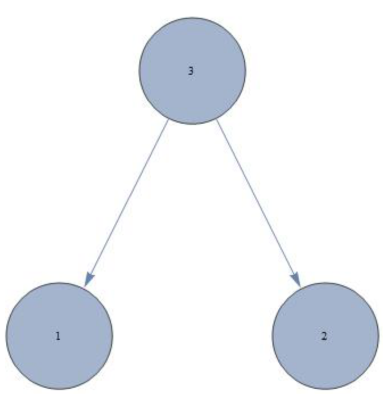
Here, we begin to look at all nine four player proper simple games. The first is simply the four player unanimity game which is clearly $\mu_A = \frac{1}{4}$, $\mu_B = \frac{1}{4}$, $\mu_C = \frac{1}{4}$, $\mu_D = \frac{1}{4}$.

h. $\mathcal{W}^{\min} = \{ABC,\ ABD\}^{+}$ implies $\mu(ABCD, M) = \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{6}, \frac{1}{6}\right)$

In this game, the $\frac{1}{2}, \frac{1}{4}, \frac{1}{4}$ kind of structure is impossible because A and B are clearly equals because they are both in all winning coalitions. B would not accept taking less than A and vice-versa. Additionally, C or D could not get $\frac{1}{2}$ because A and B are in all winning coalitions that C and D are in and more.

$\mu_A = \mu_B \geq \mu_C = \mu_D$.

#	Bloc	Type	Bloc MWCs	New full MWCs	Powers	Dominates	Dominated By
1	ABC	MinWin	ABC	ABC	$\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, 0$	3	-
2	ABD	MinWin	ABD	ABD	$\frac{1}{3}, \frac{1}{3}, 0, \frac{1}{3}$	3	-
3	CD	Block	CD	ABCD	$\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}$	-	1, 2

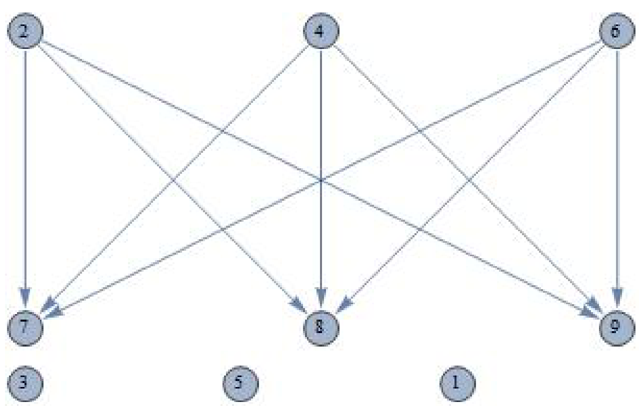


The CD bloc is dominated by structures 1 and 2 because all players in the new bloc could get $\frac{1}{3}$ instead of $\frac{1}{4}$. So, 3 is unstable and 1 and 2 are averaged to yield $\mu_A = \frac{1}{3}$, $\mu_B = \frac{1}{3}$, $\mu_C = \frac{1}{6}$, $\mu_D = \frac{1}{6}$.

i. $\mathcal{W}^{\min} = \{\text{ABC}, \text{ABD}, \text{ACD}\}^+$ implies $\mu(\text{ABCD}, M) = \left(\frac{1}{3}, \frac{2}{9}, \frac{2}{9}, \frac{2}{9}\right)$

$\mu_A \geq \mu_B = \mu_C = \mu_D$.

#	Bloc	Type	Bloc MWCs	New full MWCs	Powers	Dominates	Dominated By
1	ABC	MinWin	ABC	ABC	$\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, 0$	-	-
2	ABC	MinWin	AB, AC	AB, AC	$\frac{1}{2}, \frac{1}{4}, \frac{1}{4}, 0$	-	7 - 9
3	ABD	MinWin	ABD	ABD	$\frac{1}{3}, \frac{1}{3}, 0, \frac{1}{3}$	-	-
4	ABD	MinWin	AB, AD	AB, AD	$\frac{1}{2}, \frac{1}{4}, 0, \frac{1}{4}$	-	7 - 9
5	ACD	MinWin	ACD	ACD	$\frac{1}{3}, 0, \frac{1}{3}, \frac{1}{3}$	-	-
6	ACD	MinWin	AC, AD	AC, AD	$\frac{1}{2}, 0, \frac{1}{4}, \frac{1}{4}$	-	7 - 9
7	BC	Block	BC	ABC	$\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, 0$	2, 4, 6	-
8	BD	Block	BD	ABD	$\frac{1}{3}, \frac{1}{3}, 0, \frac{1}{3}$	2, 4, 6	-
9	CD	Block	CD	ACD	$\frac{1}{3}, 0, \frac{1}{3}, \frac{1}{3}$	2, 4, 6	-



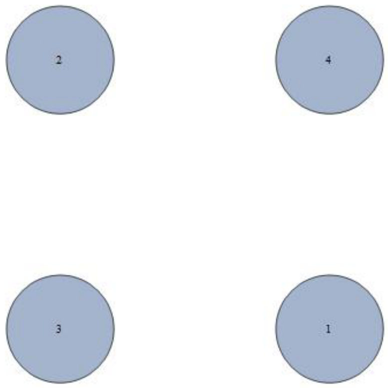
Structures 1, 3, 5, 7, 8, and 9 are the only stable structures so they are averaged, yielding $\mu_A = \frac{1}{3}$, $\mu_B = \frac{2}{9}$, $\mu_C = \frac{2}{9}$, $\mu_D = \frac{2}{9}$.

j. $\mathcal{W}^{\min} = \{\text{ABC}, \text{ABD}, \text{ACD}, \text{BCD}\}^+$ implies $\mu(\text{ABCD}, M) = \left(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}\right)$

In this game there is no player in every winning coalition so blocking blocs are not considered.

$\mu_A = \mu_B = \mu_C = \mu_D$.

#	Bloc	Type	Bloc MWCs	New full MWCs	Powers	Dominates	Dominated By
1	ABC	MinWin	ABC	ABC	$\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, 0$	-	-
2	ABD	MinWin	ABD	ABD	$\frac{1}{3}, \frac{1}{3}, 0, \frac{1}{3}$	-	-
3	ACD	MinWin	ACD	ACD	$\frac{1}{3}, 0, \frac{1}{3}, \frac{1}{3}$	-	-
4	BCD	MinWin	BCD	BCD	$\frac{1}{3}, 0, \frac{1}{3}, \frac{1}{3}$	-	-

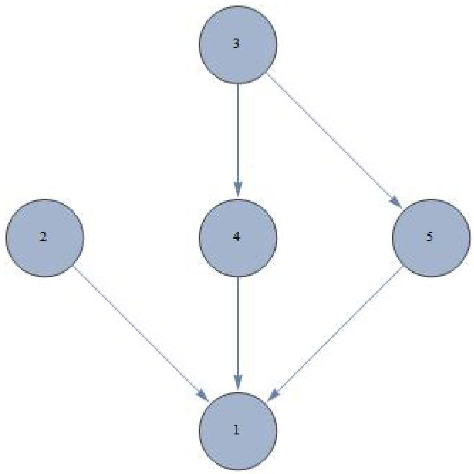


All four structures are undominated and thus stable. $\mu_A = \frac{1}{4}$, $\mu_B = \frac{1}{4}$, $\mu_C = \frac{1}{4}$, $\mu_D = \frac{1}{4}$.

k. $\mathcal{W}^{\min} = \{AB,\ ACD\}^{+}$ implies $\mu(ABCD, M) = \left(\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{8}\right)$

$\mu_A \geq \mu_B \geq \mu_C = \mu_D$.

#	Bloc	Type	Bloc MWCs	New full MWCs	Powers	Dominates	Dominated By
1	AB	MinWin	AB	AB	$\frac{1}{2}, \frac{1}{2}, 0, 0$	2, 4, 5	-
2	ACD	MinWin	ACD	ACD	$\frac{1}{3}, 0, \frac{1}{3}, \frac{1}{3}$	-	1
3	ACD	MinWin	AC, AD	AC, AD	$\frac{1}{2}, 0, \frac{1}{4}, \frac{1}{4}$	-	4, 5
4	BC	Block	BC	ABC	$\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, 0$	3	1
5	BD	Block	BD	ABD	$\frac{1}{3}, \frac{1}{3}, 0, \frac{1}{3}$	3	1



Structure 1 is undominated so it is clearly stable. However, structure 3 is also stable, because although C or D could gain power by moving to structure 4 or 5, B would promptly abandon them and move to structure 1 where both C and D get no power. Therefore, C and D would stay at structure 3. Averaging 1 and 3 we get $\mu_A = \frac{1}{2}$, $\mu_B = \frac{1}{4}$, $\mu_C = \frac{1}{8}$, $\mu_D = \frac{1}{8}$.

l. $\mathcal{W}^{\min} = \{AB,\ ACD,\ BCD\}^+$ implies $\mu(ABCD, M) = \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{6}, \frac{1}{6}\right)$

$\mu_A = \mu_B \geq \mu_C = \mu_D$.

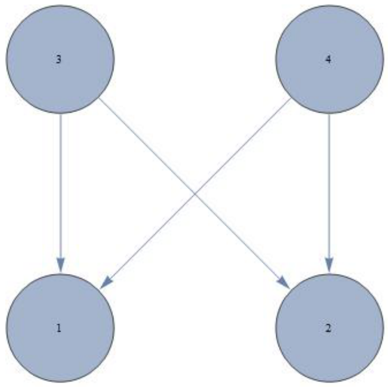
#	Bloc	Type	Bloc MWCs	New full MWCs	Powers	Dominates	Dominated By
1	AB	MinWin	AB	AB	$\frac{1}{2}, \frac{1}{2}, 0, 0$	2, 4	–
2	ACD	MinWin	ACD	ACD	$\frac{1}{3}, 0, \frac{1}{3}, \frac{1}{3}$	5	1
3	ACD	MinWin	AC, AD	AC, AD	$\frac{1}{2}, 0, \frac{1}{4}, \frac{1}{4}$	–	4
4	BCD	MinWin	BCD	BCD	$0, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}$	3	1
5	BCD	MinWin	BC, BD	BC, BD	$0, \frac{1}{2}, \frac{1}{4}, \frac{1}{4}$	–	2

1 is undominated so it is stable. 2 and 4 are unstable because they are dominated by a stable structure. Structures 3 and 5 are dominated by only unstable structures, so they are stable. The average of 1, 3, and 5 is $\mu_A = \frac{1}{3}$, $\mu_B = \frac{1}{3}$, $\mu_C = \frac{1}{6}$, $\mu_D = \frac{1}{6}$.

m. $\mathcal{W}^{\min} = \{AB,\ AC,\ BCD\}^+$ implies $\mu(ABCD, M) = \left(\frac{1}{2}, \frac{1}{4}, \frac{1}{4}, 0\right)$

$\mu_A \geq \mu_B = \mu_C \geq \mu_D$.

#	Bloc	Type	Bloc MWCs	New full MWCs	Powers	Dominates	Dominated By
1	AB	MinWin	AB	AB	$\frac{1}{2}, \frac{1}{2}, 0, 0$	3, 4	–
2	AC	MinWin	AC	AC	$\frac{1}{2}, 0, \frac{1}{2}, 0$	3, 4	–
3	BCD	MinWin	BCD	BCD	$0, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}$	–	1, 2
4	BCD	MinWin	BCD	BCD	$0, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}$	–	1, 2

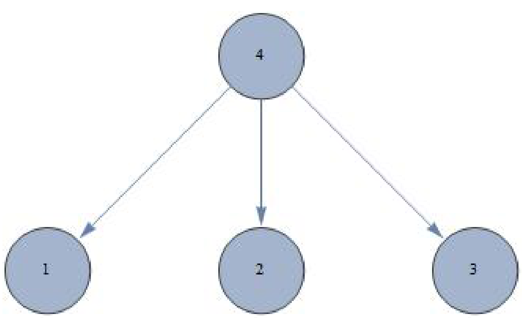


1 and 2 are stable, 3 and 4 are unstable. $\mu_A = \frac{1}{2}$, $\mu_B = \frac{1}{4}$, $\mu_C = \frac{1}{4}$, $\mu_D = 0$. Perhaps counter intuitively, D gets no power despite being in a minimal winning coalition. However, A can offer B and C $\frac{1}{2}$ of the power. The only way a BCD bloc could compete would be for D to give each B and C half of the power, but this again leaves D with none, which doesn't help D at all.

n. $\mathcal{W}^{\min} = \{AB,\ AC,\ AD\}^{+}$ implies $\mu(ABCD, M) = \left(\frac{1}{2}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}\right)$

$\mu_A \geq \mu_B = \mu_C = \mu_D$.

#	Bloc	Type	Bloc MWCs	New full MWCs	Powers	Dominates	Dominated By
1	AB	MinWin	AB	AB	$\frac{1}{2}, \frac{1}{2}, 0, 0$	4	–
2	AC	MinWin	AC	AC	$\frac{1}{2}, 0, \frac{1}{2}, 0$	4	–
3	AD	MinWin	AD	AD	$\frac{1}{2}, 0, 0, \frac{1}{2}$	4	–
4	BCD	Blocking	BCD	ABCD	$\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}$	–	1 – 3

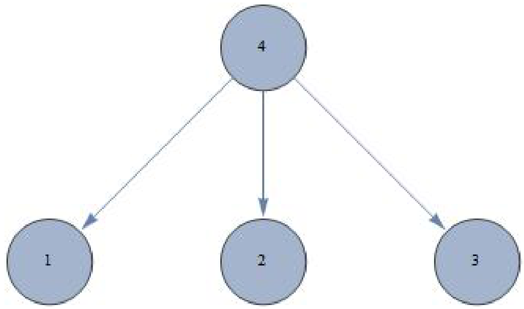


1, 2, and 3 are averaged to yield the answer $\mu_A = \frac{1}{2}$, $\mu_B = \frac{1}{6}$, $\mu_C = \frac{1}{6}$, $\mu_D = \frac{1}{6}$.

o. $\mathcal{W}^{\min} = \{AB,\ AC,\ AD,\ BCD\}^{+}$ implies $\mu(ABCD, M) = \left(\frac{1}{2}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}\right)$

$\mu_A \geq \mu_B = \mu_C = \mu_D$.

#	Bloc	Type	Bloc MWCs	New full MWCs	Powers	Dominates	Dominated By
1	AB	MinWin	AB	AB	$\frac{1}{2}, \frac{1}{2}, 0, 0$	4	–
2	AC	MinWin	AC	AC	$\frac{1}{2}, 0, \frac{1}{2}, 0$	4	–
3	AD	MinWin	AD	AD	$\frac{1}{2}, 0, 0, \frac{1}{2}$	4	–
4	BCD	MinWin	BCD	BCD	$0, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}$	–	1 – 3



The BCD bloc gains more power than in the previous game, but ultimately fails to compete with structures 1, 2, and 3. $\mu_A = \frac{1}{2}$, $\mu_B = \frac{1}{6}$, $\mu_C = \frac{1}{6}$, $\mu_D = \frac{1}{6}$.

Proofs

No player can ever have greater than $\frac{1}{2}$ power.

5. Problems and Paradoxes

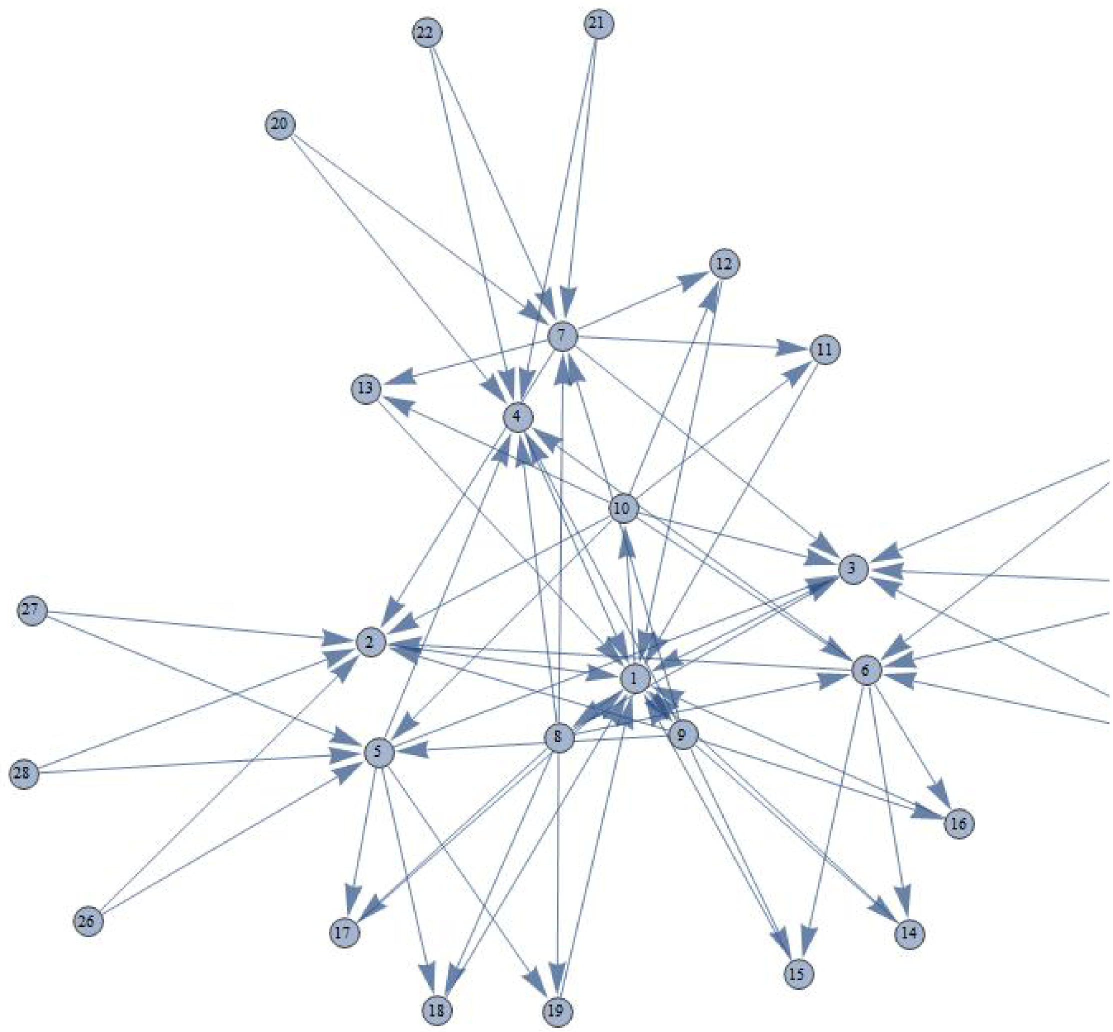
$\mathcal{W}^{\min} = \{$ABC, ADEF, BDEF, CDEF, ABDE, ABDF, ABEF, ACDE, ACDF, ACEF, BCDE, BCDF, BCEF$\}^+$ implies

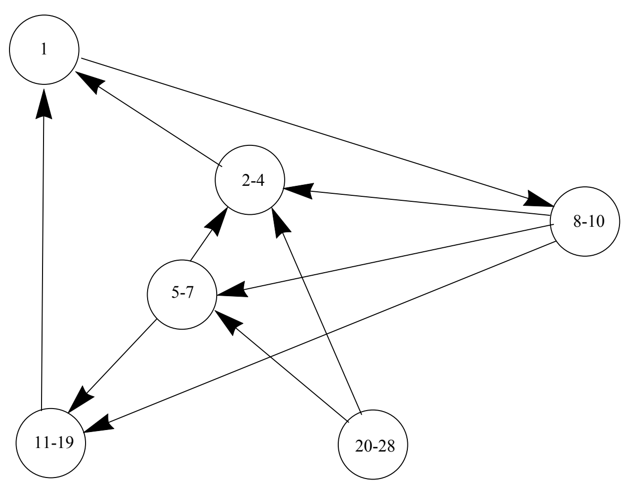
$\mu(\text{ABCDEF}, M) = \left(\frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}\right)$

The game with minimal winning coalitions ABC, ADEF, BDEF, CDEF, ABDE, ABDF, ABEF, ACDE, ACDF, ACEF, BCDE, BCDF, BCEF has no undominated structures.

$\mu_A = \mu_B = \mu_C \geq \mu_D = \mu_E = \mu_F$.

#	Bloc	Type	Bloc MWCs	New full MWCs	Powers	Dominates
1	ABC	MinWin	ABC	ABC	$\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, 0, 0, 0$	2 – 4, 11 – 19
2	ADEF	MinWin	ADEF	ADEF	$\frac{1}{4}, 0, 0, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}$	6, 7, 9, 10, 26 – 28
3	BDEF	MinWin	BDEF	BDEF	$0, \frac{1}{4}, 0, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}$	5, 7, 8, 10, 23 – 25
4	CDEF	MinWin	CDEF	CDEF	$0, 0, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}$	5, 6, 8, 9, 20 – 22
5	ADEF	MinWin	ADE, ADF, AEF	ADE, ADF, AEF	$\frac{1}{3}, 0, 0, \frac{2}{9}, \frac{2}{9}, \frac{2}{9}$	9, 10, 26 – 28
6	BDEF	MinWin	BDE, BDF, BEF	BDE, BDF, BEF	$0, \frac{1}{3}, 0, \frac{2}{9}, \frac{2}{9}, \frac{2}{9}$	8, 10, 23 – 25
7	CDEF	MinWin	CDE, CDF, CEF	CDE, CDF, CEF	$0, 0, \frac{1}{3}, \frac{2}{9}, \frac{2}{9}, \frac{2}{9}$	8, 9, 20 – 22
8	ADEF	MinWin	AD, AE, AF	AD, AE, AF	$\frac{1}{2}, 0, 0, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}$	1
9	BDEF	MinWin	BD, BE, BF	BD, BE, BF	$0, \frac{1}{2}, 0, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}$	1
10	CDEF	MinWin	CD, CE, CF	CD, CE, CF	$0, 0, \frac{1}{2}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}$	1
11	ABDE	MinWin	ABDE	ABDE	$\frac{1}{4}, \frac{1}{4}, 0, \frac{1}{4}, \frac{1}{4}, 0$	7, 10
12	ABDF	MinWin	ABDF	ABDF	$\frac{1}{4}, \frac{1}{4}, 0, \frac{1}{4}, 0, \frac{1}{4}$	7, 10
13	ABEF	MinWin	ABEF	ABEF	$\frac{1}{4}, \frac{1}{4}, 0, 0, \frac{1}{4}, \frac{1}{4}$	7, 10
14	ACDE	MinWin	ACDE	ACDE	$\frac{1}{4}, 0, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, 0$	6, 9
15	ACDF	MinWin	ACDF	ACDF	$\frac{1}{4}, 0, \frac{1}{4}, \frac{1}{4}, 0, \frac{1}{4}$	6, 9
16	ACEF	MinWin	ACEF	ACEF	$\frac{1}{4}, 0, \frac{1}{4}, 0, \frac{1}{4}, \frac{1}{4}$	6, 9
17	BCDE	MinWin	BCDE	BCDE	$0, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, 0$	5, 8
18	BCDF	MinWin	BCDF	BCDF	$0, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, 0, \frac{1}{4}$	5, 8
19	BCEF	MinWin	BCEF	BCEF	$0, \frac{1}{4}, \frac{1}{4}, 0, \frac{1}{4}, \frac{1}{4}$	5, 8
20	ABDE	MinWin	ABD, ABE	ABD, ABE	$\frac{1}{3}, \frac{1}{3}, 0, \frac{1}{6}, \frac{1}{6}, 0$	–
21	ABDF	MinWin	ABD, ABF	ABD, ABF	$\frac{1}{3}, \frac{1}{3}, 0, \frac{1}{6}, 0, \frac{1}{6}$	–
22	ABEF	MinWin	ABE, ABF	ABE, ABF	$\frac{1}{3}, \frac{1}{3}, 0, 0, \frac{1}{6}, \frac{1}{6}$	–
23	ACDE	MinWin	ACD, ACE	ACD, ACE	$\frac{1}{3}, 0, \frac{1}{3}, \frac{1}{6}, \frac{1}{6}, 0$	–
24	ACDF	MinWin	ACD, ACF	ACD, ACF	$\frac{1}{3}, 0, \frac{1}{3}, \frac{1}{6}, 0, \frac{1}{6}$	–
25	ACEF	MinWin	ACE, ACF	ACE, ACF	$\frac{1}{3}, 0, \frac{1}{3}, 0, \frac{1}{6}, \frac{1}{6}$	–
26	BCDE	MinWin	BCD, BCE	BCD, BCE	$0, \frac{1}{3}, \frac{1}{3}, \frac{1}{6}, \frac{1}{6}, 0$	–
27	BCDF	MinWin	BCD, BCF	BCD, BCF	$0, \frac{1}{3}, \frac{1}{3}, \frac{1}{6}, 0, \frac{1}{6}$	–
28	BCEF	MinWin	BCE, BCF	BCE, BCF	$0, \frac{1}{3}, \frac{1}{3}, 0, \frac{1}{6}, \frac{1}{6}$	–





This second graph is perhaps an oversimplification, because, for example, 5 does not dominate every-thing 20-28. However, it is far easier to see what is going on here than in the graph above where all 28 structures are included. This particular game presents a problem because there are no undominated structures. Fortunately, the Von Neumann-Morgenstern labeling process still works for this problem through a guess and check method.

Proof of Stability:

Suppose 1 is stable. Then 2-4, 8-19 are unstable. Then 5, 6, and 7 are dominated only by 2-4, 11-19, all of which are unstable. Therefore, 5-7 are stable. Now, 20-28 are unstable because each of them is dominated by a stable structure, either 5, 6, or 7. Then, {1, 5, 6, 7} is one possible stable set.

Now suppose 1 is not stable and at least one of 2-4 is stable. Without loss of generality, suppose 2 is stable. Then 1 must be unstable. Then at least one of 8-10 must be stable. Then at least three of 11-19 must be unstable. But the only thing that dominates 11-19 is 1. Since 1 is unstable, 11-19 must be stable. 11-19 cannot be both stable and unstable.

Now suppose 1-4 are not stable and at least one of 5-7 is stable. Without loss of generality, suppose 5. Then 3, 4, 9, 10, 17-19, 26-28 are unstable. 3 and 4 must be dominated by a stable structure, so 1 is stable, a contradiction.

Suppose 1-7 are not stable and at least one of 8-10 is stable. Without loss of generality, suppose 8. Then 1 and 17-19 are unstable. But, 17-19 are only dominated by 1, which implies a contradiction.

Suppose 1-10 are not stable and at least one of 11-19 is stable. Without loss of generality, suppose 11. Then 1, 7, 10 must be unstable. If 1 is unstable, 2-4 must be stable. But this implies that 8-10 must be unstable. But 1 must be dominated by something stable and it is only dominated by 8-10. Therefore, 11-19 cannot be stable.

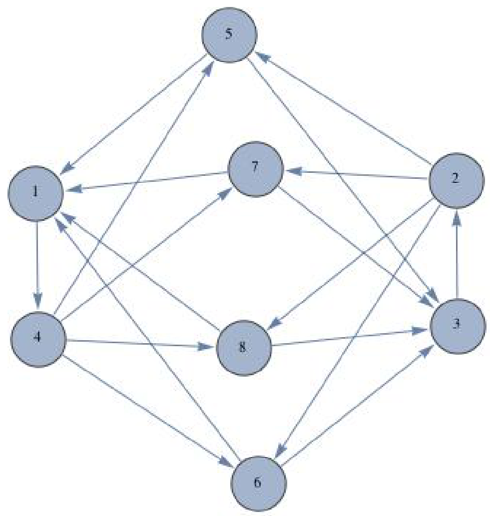
Suppose 1-10 are not stable and at least one of 20-28 is stable. Then 2-7 are unstable. Since 2-4 are unstable, 1 must be stable. Since 1 is stable, 11-19 must be unstable. But then 5-7 are dominated by 2-4, 11-19, all of which are unstable. This is a contradiction.

Therefore, {1, 5, 6, 7} is the only stable set for the game [9; 3, 3, 3, 2, 2, 2]. By averaging the powers of the games 1, 5, 6, 7 we get the answer $A = \frac{1}{6}$, $B = \frac{1}{6}$, $C = \frac{1}{6}$, $D = \frac{1}{6}$, $E = \frac{1}{6}$, $F = \frac{1}{6}$. It could be argued that A, B, and C clearly have more power than D, E, and F and thus should get more power. That issue could be resolved by weighting 1, 5, 6, and 7 in some different, but reasonable, way.

The Triangle Problem: $\mathcal{W}^{\min} = \{\text{ABCD, ABEF}\}^{+}$

$\mu_A = \mu_B \geq \mu_C = \mu_D = \mu_E = \mu_F$.

#	Bloc	Type	Bloc MWCs	New full MWCs	Powers	Dominates	Dominat
1	ABCD	MinWin	ABCD	ABCD	$\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, 0, 0$	5 – 8	4
2	ABCD	MinWin	AB, ACD, BCD	AB, ACD, BCD	$\frac{1}{3}, \frac{1}{3}, \frac{1}{6}, \frac{1}{6}, 0, 0$	3	5 –
3	ABEF	MinWin	ABEF	ABEF	$\frac{1}{4}, \frac{1}{4}, 0, 0, \frac{1}{4}, \frac{1}{4}$	5 – 8	2
4	ABEF	MinWin	AB, AEF, BEF	AB, AEF, BEF	$\frac{1}{3}, \frac{1}{3}, 0, 0, \frac{1}{6}, \frac{1}{6}$	1	5 –
5	CDE	Blocking	CDE	ABCDE	$\frac{1}{5}, \frac{1}{5}, \frac{1}{5}, \frac{1}{5}, \frac{1}{5}, 0$	2, 4	1,
6	CDF	Blocking	CDF	ABCDF	$\frac{1}{5}, \frac{1}{5}, \frac{1}{5}, \frac{1}{5}, 0, \frac{1}{5}$	2, 4	1,
7	CEF	Blocking	CEF	ABCEF	$\frac{1}{5}, \frac{1}{5}, \frac{1}{5}, 0, \frac{1}{5}, \frac{1}{5}$	2, 4	1,
8	DEF	Blocking	DEF	ABDEF	$\frac{1}{5}, \frac{1}{5}, 0, \frac{1}{5}, \frac{1}{5}, \frac{1}{5}$	2, 4	1,



In this game, it is clear that there are no undominated structures. However, unlike the last game, it is impossible to use the Von Neumann-Morgenstern labeling ideas without running into a contradiction.

Proof:

Suppose 1 or 3 is stable. Suppose 1. Then 4, 5-8 must be unstable. 4 is dominated only by 5-8, all of which are now unstable. An unstable structure cannot be dominated only by unstable structures. Therefore, 1 and 3 cannot be stable.

Now assume that 1 and 3 are unstable and 2 or 4 is stable. Assume 2, without loss of generality. Then 1, 3, 5-8 are unstable. 5-8 are dominated by 1 and 3. 3 and 1 are unstable, but 5-8 must be dominated by a stable structure, a contradiction.

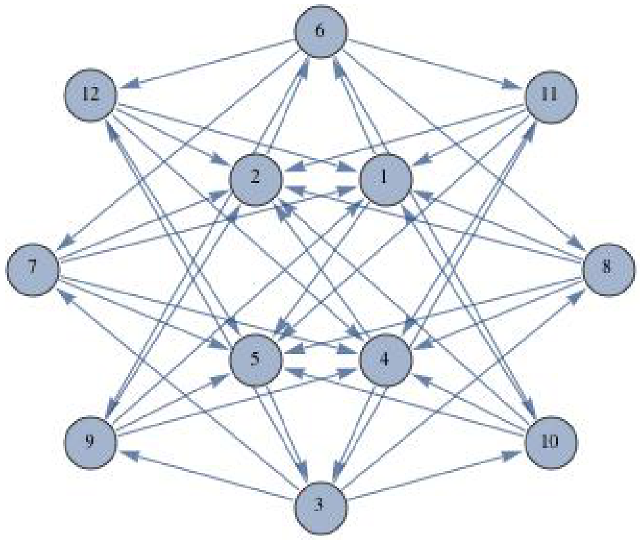
Suppose 1-4 are unstable and at least one of 5-8 is stable. Suppose 5, without loss of generality. But 3 is dominated only by 2. 3 and 2 cannot both be unstable. Therefore 5-8 cannot be stable.

The Von Neumann-Morgenstern labeling does not work on this graph. Since it isn't clear what is stable and unstable, one could imagine simply averaging the eight structures in some manner. If each structure is assigned $\frac{1}{8}$ probability, $\mu_A = \frac{59}{240}$, $\mu_B = \frac{59}{240}$, $\mu_C = \frac{61}{480}$, $\mu_D = \frac{61}{480}$, $\mu_E = \frac{61}{480}$, $\mu_F = \frac{61}{480}$. However, now we have violated monotonicity. The four player unanimity game ABCD would clearly give each player $\frac{1}{4}$ of the power. If we add the minimal winning coalition ABEF, A and B lose power, $\frac{1}{4} > \frac{59}{240}$, if the eight above structures are evenly averaged.

Another Triangle: $\mathcal{W}^{\min} = \{ABCD, AEFG\}^{+}$

$\mu_A \geq \mu_B = \mu_C = \mu_D = \mu_E = \mu_F = \mu_G$.

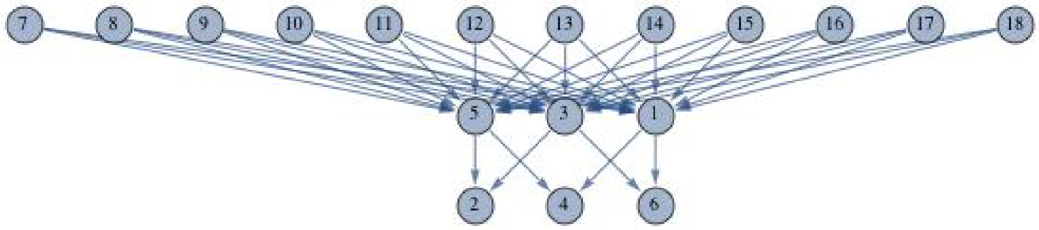
#	Bloc	Type	Bloc MWCs	New full MWCs	Powers	Dominates	Dc
1	ABCD	MinWin	ABCD	ABCD	$\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, 0, 0, 0$	7 – 12	
2	ABCD	MinWin	ABC, ABD, ACD	ABC, ABD, ACD	$\frac{1}{3}, \frac{2}{9}, \frac{2}{9}, \frac{2}{9}, 0, 0, 0$	4, 7 – 12	
3	ABCD	MinWin	AB, AC, AD	AB, AC, AD	$\frac{1}{2}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}, 0, 0, 0$	4, 5	
4	ABEF	MinWin	AEFG	AEFG	$\frac{1}{4}, 0, 0, 0, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}$	7 – 12	
5	ABEF	MinWin	AEF, AEG, AFG	AEF, AEG, AFG	$\frac{1}{3}, 0, 0, 0, \frac{2}{9}, \frac{2}{9}, \frac{2}{9}$	1, 7 – 12	
6	ABEF	MinWin	AE, AF, AG	AE, AF, AG	$\frac{1}{2}, 0, 0, 0, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}$	1, 2	
7	BCDE	Blocking	BCDE	ABCDE	1/[illegible], 1/[illegible], 1/[illegible], 1/[illegible], 1/[illegible], 0, 0	3, 6	[illegible]



Another Monotone Violation: {ABCD, ABEF, ABGH}$^+$

$\mu_A = \mu_B \geq \mu_C = \mu_D = \mu_E = \mu_F = \mu_G$.

#	Bloc	Type	Bloc MWCs	New full MWCs	Powers	Dominates
1	ABCD	MinWin	ABCD	ABCD	$\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, 0, 0, 0, 0$	7 – 18
2	ABCD	MinWin	AB, ACD, BCD	AB, ACD, BCD	$\frac{1}{3}, \frac{1}{3}, \frac{1}{6}, \frac{1}{6}, 0, 0, 0, 0$	3, 5
3	ABEF	MinWin	ABEF	ABEF	$\frac{1}{4}, \frac{1}{4}, 0, 0, \frac{1}{4}, \frac{1}{4}, 0, 0$	7 – 18
4	ABEF	MinWin	AB, AEF, BEF	AB, AEF, BEF	$\frac{1}{3}, \frac{1}{3}, 0, 0, \frac{1}{6}, \frac{1}{6}, 0, 0$	1, 5
5	ABGH	MinWin	ABGH	ABGH	$\frac{1}{4}, \frac{1}{4}, 0, 0, 0, 0, \frac{1}{4}, \frac{1}{4}$	7 – 18
6	ABGH	MinWin	AB, AGH, BGH	AB, AGH, BGH	$\frac{1}{3}, \frac{1}{3}, 0, 0, 0, 0, \frac{1}{6}, \frac{1}{6}$	1, 3
7	CDEG	Blocking	CDEG	ABCDEG	$\frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}, 0, \frac{1}{6}, 0$	–
8	CDEH	Blocking	CDEH	ABCDEH	$\frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}, 0, 0, \frac{1}{6}$	–
9	CDFG	Blocking	CDFG	ABCDFG	$\frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}, 0, \frac{1}{6}, \frac{1}{6}, 0$	–
10	CDFH	Blocking	CDFH	ABCDFH	$\frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}, 0, \frac{1}{6}, 0, \frac{1}{6}$	–
11	CEFG	Blocking	CEFG	ABCEFG	$\frac{1}{6}, \frac{1}{6}, \frac{1}{6}, 0, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}, 0$	–
12	CEFH	Blocking	CEFH	ABCEFH	$\frac{1}{6}, \frac{1}{6}, \frac{1}{6}, 0, \frac{1}{6}, \frac{1}{6}, 0, \frac{1}{6}$	–
13	DEFG	Blocking	DEFG	ABDEFG	$\frac{1}{6}, \frac{1}{6}, 0, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}, 0$	–
14	DEFH	Blocking	DEFH	ABDEFH	$\frac{1}{6}, \frac{1}{6}, 0, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}, 0, \frac{1}{6}$	–
15	CEGH	Blocking	CEGH	ABCEGH	$\frac{1}{6}, \frac{1}{6}, \frac{1}{6}, 0, \frac{1}{6}, 0, \frac{1}{6}, \frac{1}{6}$	–
16	CFGH	Blocking	CFGH	ABCFGH	$\frac{1}{6}, \frac{1}{6}, \frac{1}{6}, 0, 0, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}$	–
17	DEGH	Blocking	DEGH	ABDEGH	$\frac{1}{6}, \frac{1}{6}, 0, \frac{1}{6}, \frac{1}{6}, 0, \frac{1}{6}, \frac{1}{6}$	–
18	DFGH	Blocking	DFGH	ABDFGH	$\frac{1}{6}, \frac{1}{6}, 0, \frac{1}{6}, 0, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}$	–



Here, 2, 4, 6 are stable because they are undominated. So, 1, 3, and 5 are unstable. But then 7-18 are stable. However, A and B get only $\frac{1}{5}$ of the power if alll these stable structures are weighted equally and then averaged. This averaging scheme violates monotonicity, because a four player unanimity game gives A and B $\frac{1}{4}$. A and B have added ABEF and ABGH as new minimal winning coalitions. If anything, their power should increase.

6. Future Work

At this juncture, much remains up in the air about the Machiavelli Index. There is plenty of room for future work. Since the current definition of the Machiavelli Power Index violates monotonicity, one approach might be to make adjustments to it in the hopes of satisfying and proving monotonicity. Our definition of stability may need to be redefined to fit games like ABCD, ABEF. Additionally, it may be true that our current definition allows for multiple stable sets. It is not clear how those should be averaged should that situation arise. Or perhaps, rather than weighting all stable structures equally in an average, perhaps a reasonable weighting scheme can be discovered.

Non-minimal winning blocs and blocking blocs that allow other blocking blocs could be added to the problematic games from the previous section. Along a similar vein, blocs with dummies should be considered (this has not been done for any number of players). Bloc structures that have power inconsistencies, too, should be considered. Our hope and our intuition is that these structures are always unstable and should not factor into the average at all, however, this is far from proven.

A final idea for future work is to calculate games with more than four players. To contemplate any game with minimal winning coalitions of five or more players, one needs to know all possible power divisibility schemes for that many players.

Acknowledgments

I would like to thank David Housman for his mentoring and guidance. Without him, this paper would not have been possible.

I would also like to thank Goshen College and its Maple Scholars program which provided funding for this project.

References

Felsenthal, Dan S., and Mosh Machover. “Postulates and Paradoxes of Relative Voting Power ? A Critical
Re-appraisal.” Theory and Decision 38.2 (1995): 195-229. Print.

Richard, Gina. Monotonic Power Indices. Goshen College. Goshen College, 10 Aug. 2008. Web. 29 July 2013.

Shapley, L. S. “Simple Games: An Outline of the Descriptive Theory.” Behavioral Science 7.1 (1962): 59-66. Print.