Properties of Simple Games

John Baldan

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Simple games are the class of cooperative games in characteristic function form for which $v(S)=1$ or $v(S)=0$ for all coalitions S of the set of players N. In words, every coalition is either "winning" (all-powerful) or "losing" (ineffectual). Simple games can be used to model the process of voting in parliamentary or other political bodies. When a motion is proposed, the coalition S that votes in favor of the proposal either carries enough weight under the existing voting scheme to enable the proposal to pass ($v(S)=1$) or it does not ($v(S)=0$).

Formally, a simple game is a pair (N, W) where $N=\{1,2,...,n\}$ is a set of players and W is the set of coalitions (subsets of N) which are winning. There are three conditions on W:

1. $N \in W$
2. $\emptyset \notin W$
3. If $T \in W$ and $S \supseteq T$ then $S \in W$.

Each condition is necessary if the simple game is to adequately model real-life voting: unanimity for or against a proposal is decisive; and adding players to a winning coalition (or removing them from a losing coalition) should not affect the outcome of a vote.

A simple game can be represented more compactly as (N, M) where M is the set of minimal winning coalitions. A minimal winning coalition is a winning coalition which contains no proper subset that is also winning. Thus the set M of minimal winning coalitions of a game does not contain two coalitions, one of which is a subset of the other. Such a set of sets is known as a clutter. One can easily derive W by appending players to each element of M. From now on, we will represent simple games with the (N, M) notation.

An interesting question to consider is: how many distinct sets of voting rules are possible for a voting body of n players? The set M uniquely determines a set of voting rules: for any coalition, it assigns the result "winning" or "losing". How many sets M are there on n players? To simplify matters, we only consider simple games of n players with no "dummies"; i.e., where each player has some say in the outcome of a vote. Thus every player must appear in at least one of the minimal winning coalitions.

When n=1, M={A} (where A is the only player) is the only possibility. When n=2, M={AB} or M = {A,B} are the two possibilities (M= {A} is not acceptable because B is a dummy). When n=3, M={A,B,C}, M= {AB,C}, M={AB,AC}, M={AB,AC,BC} and M={ABC} are the five simple games. Shapley [1] determined that there are also 20 4-player simple games, and proceeded to list all simple games of 4 or fewer players. He announced as well that there are 179 5-player simple games, which he later revised to 180. [3]