A categorization of the evolution of spatial cellular automata in symmetric binary decision games

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Abstract

Much research has been devoted to specific two person, symmetric, binary decision (STPBD) games in the spatial context of cellular neighborhoods. Nowak first proposed a spatial model for these games, in which players are located in a square lattice and play a Prisoner's Dilemma (PD) game against each of their neighbors, using either the strategy of AllD or AllC. To move the simulation into the next generation, each cell will compare its cumulative score to that of its neighbors, and adopt the most successful strategy it finds. Nowak categorized those PD games with payoffs $S = P = 0$, $R = 1$, and $T = b(b > 1)$. His research revealed different end behaviors for values of the variable b, which are mostly independent of initial conditions [2]. Further research into this model includes studying alternative STPBD games such as the Hawk-Dove game [1], using different grid structures, applying noise to the game, and changing the independent variables in the payoff matrix [4]. We aim to give a more general approach to scoring these games that will map a cell and its neighborhood deterministically and directly onto a "cumulative" score, without concerning ourselves with two person interactions. The advantages of this are in its simplicity of representation and computation as well as its flexibility in representing the payoffs of every STPBD game yet studied. It is also capable of representing payoffs that could not be represented as sums of two person interactions. This system avoids the pitfalls of "equivalency regions," payoffs which are numerically different, but are identical in simulation because of their underlying ordering [3]. Once we have defined our system, we will be equipped to classify the immediate behavior of several small configurations of cooperators and defectors located in an infinite sea of the other. This will serve as a demonstration of the usefulness of generality in this system.

1 Introduction

This paper attempts to generalize many different scoring formulas and payoff matrices for symmetric, two person, binary decision (STPBD) games (such as Prisoner's Dilemma, Chicken, Snowdrift, etc.) in the context of a neighborhood grid structure. In the simulation, cells arranged in a square grid choose either to cooperate or defect. They are then granted some score for their choice based on what their neighbors choose. Each cell makes its choice of move for the next round to mimic the most successful cell in its neighborhood.

This paper will focus on the eight-neighbor square grid, because it has been the most standard model in past research. However, the system proposed is easily applied to any k-regular grid structure. This paper is merely an introduction to and demonstration of the usefulness of generalized score orderings.

In the eight-neighbor grid, two cells are neighbors iff they share an edge or corner. For example, in Figure 1, the neighborhood of cell V is labelled $R..Z$. We say that V has 8 neighbors, and the neighborhood of V has 9 cells, which include V.

Though scoring mechanisms have usually been defined in the two person context such that a cumulative score is found for each cell through playing two person games with each of its neighbors [2][1], we can simplify and generalize the process. All possible scoring formulas are essentially a preference ordering for the different outcomes. In an 8-neighbor system, there are only 18 possible scores for a cell at the end of a generation: The cell cooperated/defected while 0 through 8 of its neighbors cooperated. We will now describe the notation to refer to these outcomes.

Definition Let C_i be either a cooperating cell with i cooperating neighbors, or the generational score of such a cell. Similarly, let D_i reference either a defecting cell with i cooperating neighbors, or the generational score of such a cell. We assume that it is always preferable for a cell for more of its neighbors to cooperate. Thus we assume in this paper that C and D are increasing sequences ($C_0 < C_1 < \cdots < C_7 < C_8$ and $D_0 < D_1 < \cdots < D_7 < D_8$). We will not always conform to the Prisoner's Dilemma principle that defection is a strictly dominating strategy, which implies $C_0 < D_0, C_1 < D_1, \cdots, C_8 < D_8$. Nor do we restrict ourselves to the principle that a cooperator surrounded by cooperators is more successful than a defector surrounded by defectors ($D_0 < C_8$), though this comparison would never actually be made by a cell (How could a cell have in its neighborhood both a defector surrounded by defectors and a cooperator surrounded by cooperators?).

Determining a unique score ordering is equivalent to ordering a list of 9 C's and 9 D's. As one example, we will derive the score ordering for a commonly used Prisoner's Dilemma system: $T = 5, R = 3, P = 1, S = 0$:

$$
\begin{aligned}
C_0 &= 8S = 0 & D_0 &= 8P = 8 \\
C_1 &= R + 7S = 3 & D_1 &= T + 7P = 12 \\
C_2 &= 2R + 6S = 6 & D_2 &= 2T + 6P = 16 \\
C_3 &= 3R + 5S = 9 & D_3 &= 3T + 5P = 20 \\
C_4 &= 4R + 4S = 12 & D_4 &= 4T + 4P = 24 \\
C_5 &= 5R + 3S = 15 & D_5 &= 5T + 3P = 28 \\
C_6 &= 6R + 2S = 18 & D_6 &= 6T + 2P = 32 \\
C_7 &= 7R + S = 21 & D_7 &= 7T + P = 36 \\
C_8 &= 8R = 24 & D_8 &= 8T = 40
\end{aligned}
$$

And here is the generalized ordering, made by simply merging C and D in order:

$$C_0 < C_1 < C_2 < D_0 < C_3 < C_4 = D_1 < C_5 < D_2 < C_6 < D_3 < C_7 < C_8 = D_4 < D_5 < D_6 < D_7 < D_8$$

An ordering such as this one contains all the information necessary to define the rules of a simulation without concerning ourselves with the numerical payoffs which it may have derived from.

What choice does a cell make when two neighbors of contrary move are tied for the top score? Nowak chose to have cells keep their current move in such a case [2]. However, to preserve the simplicity of the ordering and avoid stagnation of forms, we will resolve ties by making one of the moves dominant over the other (essentially switching the equal sign to a greater than or less than). Thus we restrict our focus to orderings where no two outcomes are equal.

2 Applying inequalities to a particular small body

We will now apply these generalizations to determine every evolution of a 3×3 square of cooperators surrounded by defectors. This is meant to be an example that will equip the reader to perform these operations either visually or mechanically with greater ease. We will determine its outcomes in a single generation of its evolution.

Example Figure 1 shows the body in question: It is a 3×3 body of cooperators surrounded by a sea of defectors. The payoffs for each cell are displayed in white text. We will first determine which scores are pivotal for its immediate evolution.

Each cell, in determining its next move, chooses the move of the highest scoring cell in its neighborhood of 9 cells (which includes itself). Since both the C and D payoff sequences are increasing, this comparison can be simplified to comparing the highest cooperator score to the highest defector score in its neighborhood. We call the comparison of those two values a *pivotal comparison*. A few cells, like cell Z in the center of the figure, can only see one move type, so it does not make any comparison, and trivially chooses to keep the same move in the next generation. Marked in Figure 2 are the pivotal comparisons made by the cells, omitting some obvious symmetries. The pivotal comparisons are as follows:

$$D_2 \overset{?}{<>} C_3$$

$$D_3 \overset{?}{<>} C_5$$

$$D_3 \overset{?}{<>} C_8$$

To know every possible evolution of this figure, we must use each combination of $<$ and $>$ in these inequalities. This would give $2^3 = 8$ possibilities. However, D_3 is compared to both C_5 and C_8. Because $C_5 < C_8$, we can eliminate any situation in which $C_8 < D_3 < C_3$. Removing this possibility from the two orderings of D_2 and C_3, we have only 6 cases.

Suppose that $C_3 < D_2$ and $(C_5 <)C_8 < D_3$. This is all the information we require to determine the next generation of the simulation. Every cell that made a pivotal comparison in Figure 2 (and their symmetrical counterparts) chooses the greater of the two, and will adopt the associated move. In this case, defection wins on every front, and so every cell making a comparison will remain/become a defector, as illustrated in the first case of Figure 2. The body of cooperators is reduced to a single cell.

Suppose on the other hand that $D_2 < C_3$ and $(C_5 <)C_8 < D_3$. In this case, we know that those cells comparing C_3 and D_2 will become cooperators (specifically R and its symmetrical equivalents), but the other comparisons will result in defector win. This will have the curious result similar to the 5 side of a die, as illustrated in the second case of Figure 2

Suppose that $C_3 < D_2$ and $C_5 < D_3 < C_8$. This will have no effect on the cells; each comparison made will be in favor of the current move. See case 3.

Suppose that $D_2 < C_3$ and $C_5 < D_3 < C_8$. Now R will be won to the cooperators, but the other cells will keep their move. See case 4.

Suppose that $C_3 < D_2$ and $D_3 < C_5(< C_8)$. Since $C_3 < D_2$, the resulting figure will have defectors at its corners (cell R). See case 5.

Suppose finally that $D_2 < C_3$ and $D_3 < C_5(< C_8)$. One could think of the comparison of C_3 and D_2 as a switch for corners. Now, as opposed to the first case, the cooperators win at every comparison, and result in a 5×5 square. See case 6.

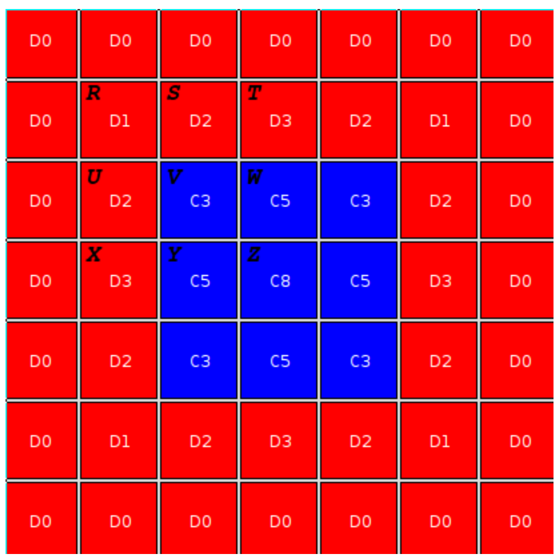

Figure 1: A 3×3 square of cooperators (blue) in a sea of defectors (red). The scores of each cell are given in the centers of each cell.

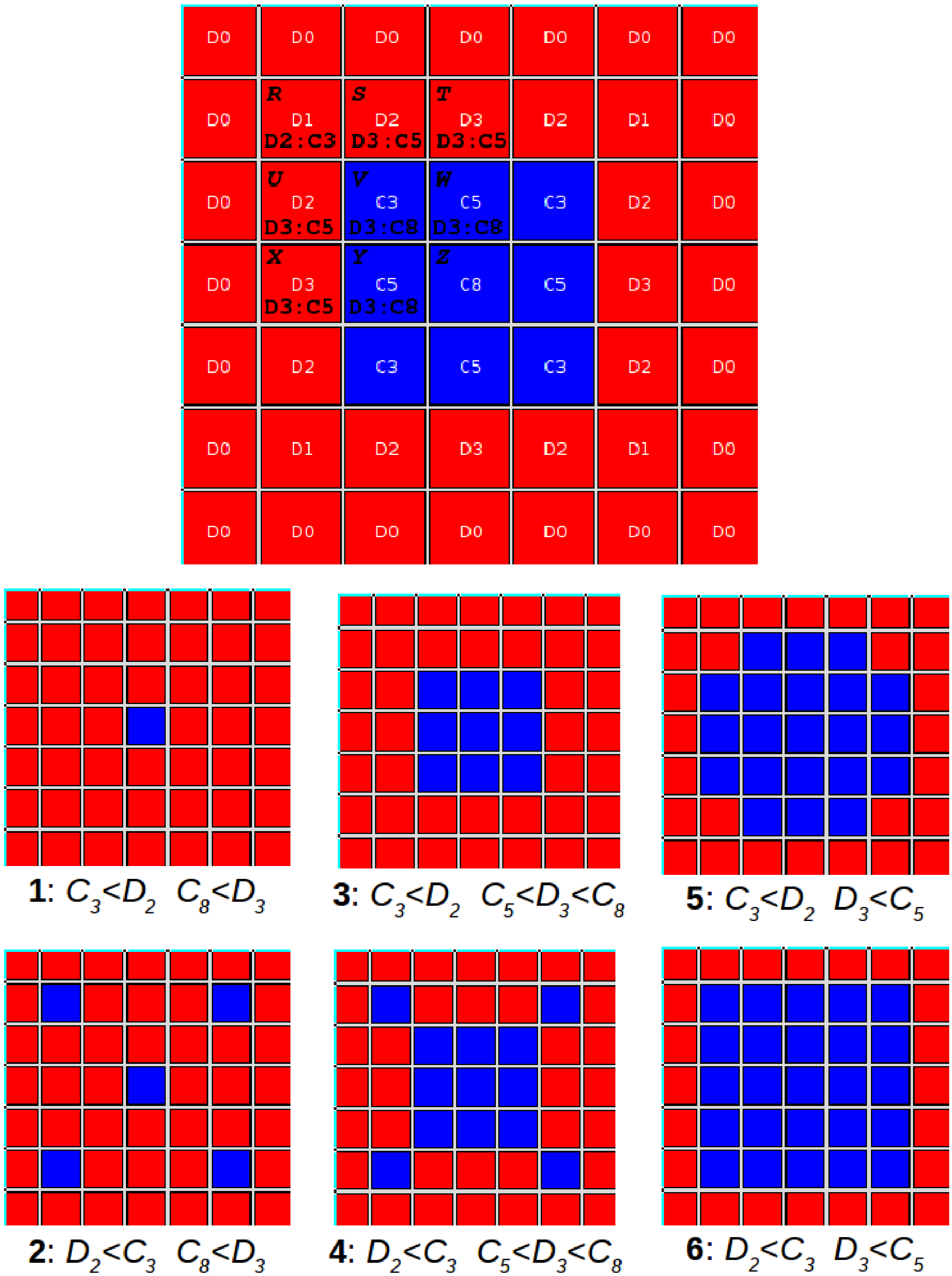

Figure 2: *Top:* This diagram of the 3x3 indicates the pivotal comparisons made by each cell, with some symmetries omitted. Some cells do not see any opposing strategy in their neighborhood, like cell Z, so no comparison is made. *Bottom:* The 6 possible outcomes for the 3x3, with respective conditions.

3 General examination of elementary bodies

The score ordering system simplifies proofs about the behaviors of elementary bodies in these simulations, because all the relevant information about scores is included in condensed form. We can diagram various outcomes without consideration of what STPBD payoff matrix may produce a situation where some C_i dominates some D_j. We will soon prove a theorem about certain bodies that assumes only a scoring where $C_4 > D_3$. Because the payoffs C and D are increasing sequences, this one inequality implies many other derivative inequalities that we don't bother to mention. We will begin with some definitions and corollaries, which are useful in understanding Figure 5.

Definition A *body* is a finite and contiguous group of cells with identical strategy. A *convex body* is a body which can be constructed by cutting off 0 to 4 corners of a rectangle, at 45°angles. A convex body could also be known as a *45°truncated rectangle.* A cell is a *neighbor* to a body iff it is neighbor to a cell in the body.

Definition An *edge* is a maximal set of two or more cells in a convex body whose centers are collinear and each of whom border at least three cells external to the body. In Figure 3, the edges are $\{S, T, U, V\}$, $\{V, Y\}$, $\{Y, Z\}$, and $\{Z, W, S\}$. A *corner* is a cell that is on two edges. In Figure 3, S, V, Y, and Z are corners. A *middle edge* is an edge excluding its two corners. An edge is *flat* if its cells are horizontally or vertically aligned. An edge is *diagonal* (stairstepping appearance) otherwise.

Corollary 3.1. *Every corner in a convex body is one of the following: connecting a diagonal edge to a straight edge at a 45°angle (See Figure 3, cell S), connecting a straight edge to a straight edge (See cell V), connecting a diagonal edge to a diagonal edge (See cell Z), or connecting a diagonal edge to a straight edge at a 135°angle (See cell Y).*

Proof. Based on our constructional definition of a convex body, no other edge types besides straight or diagonal exist. Since a type of corner is determined by the type and orientation of its two edges, we merely establish that there is only one angle at which to connect two diagonal edges or two flat edges, and two ways to connect a diagonal and straight edge. All four of these corner types are represented in Figure 3 and Figure 4. □

Corollary 3.2. *No neighboring cell to a convex body neighbors more than 3 cells inside the body.*

Proof. Note in the previous proof that Figure 3 contains all 4 corner types and all 2 edge types. Also, growing and shrinking the body does not increase the defector scores (compare to Figure 4), so we can use this as a representative figure. Because no defector is D_4 or greater in Figure 3, the corollary is true for all convex bodies. □

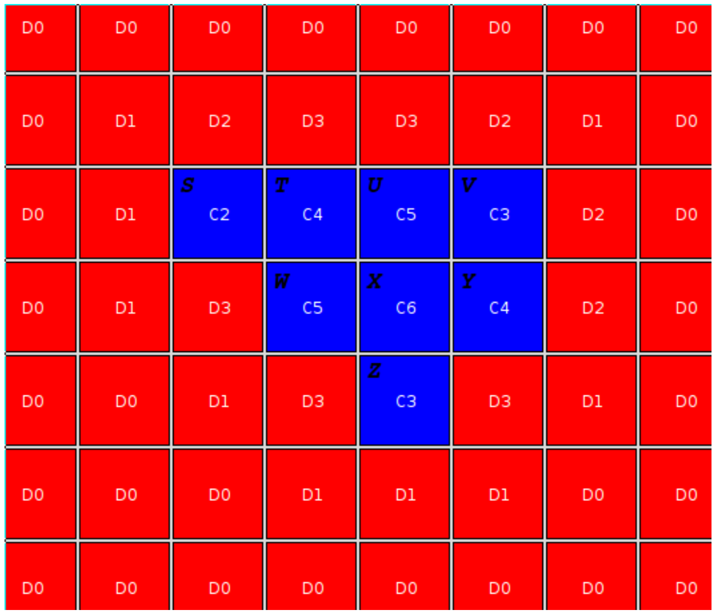

Figure 3: A representative convex body, displaying the 4 corner types and 2 edge types. Special attention should be given to those cooperating and defecting cells along the edges and corners. Note that scaling up this body does not affect the scores of corners and edges (See Figure 4).

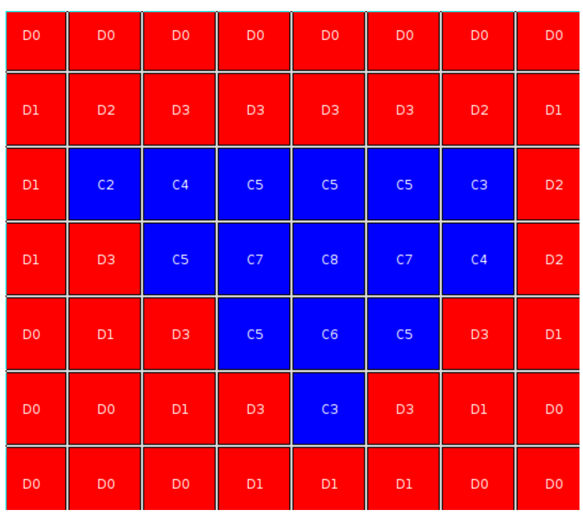

Figure 4: A similar body to Figure 3, but with longer edges on three sides. This extension only affects the number of cells in the middle edges, and not their score of C_{4+}, nor the score of any of the corner types.

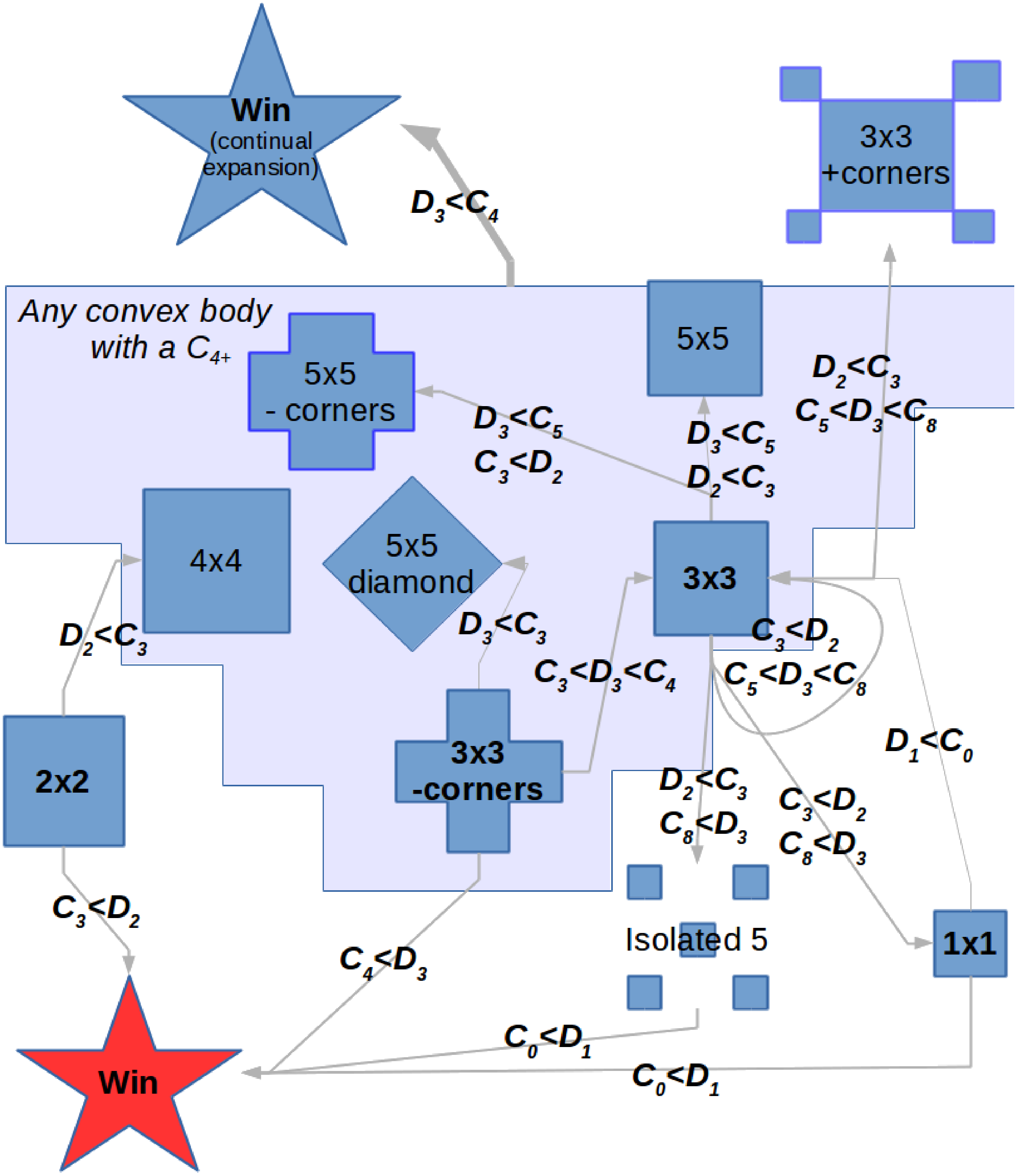

Figure 5: A general reference for small cooperator bodies in an infinite sea of defectors in an eight-neighbor system. Bold labels in a body indicate that every exit has been diagrammed.

4 Convexity and Expansion

The reference in Figure 5 makes a few general assertions about categories of bodies. The most notable of these is that the large category of convex bodies will under certain conditions grow without bound. We will walk through the proof of this theorem in order to demonstrate how similar reasoning can provide further general conclusions.

Theorem 4.1. *If B_C is a convex body of cooperators in an infinite sea of defectors, in an 8-neighbor grid system, where $C_4 > D_3$, and if B_C contains a cell of score C_4 or greater, then the body will expand without bound in convex form.*

Proof. We will show that a body satisfying the above conditions will necessarily grow while maintaining the conditions into the next generation.

Let $S(i)$ be the score earned by cell i.

Let B_C be such a body as described in the theorem, and N_D be the neighboring defectors to this body. Since B_C contains every cooperator in the plane, every defector d will have a score of D_0 if it is not in N_D, and greater than D_0 if it is in N_D. By Corollary 3.2, we establish that every cell d in N_D has score $D_0 < S(d) < D_4$, or a maximum score of D_3

Because the scores C and D are increasing sequences, C_4 or greater is a dominating score in this generation. Thus, all the neighbors of C_4 or greater will be cooperators in the next round. With this we establish that there will be growth in the figure, as every C_4 or greater becomes a C_8 in the next generation.

The corners of B_C will either be C_4 (forming a 135°angle), C_3 (90°), or C_2 (45°). In each of these cases, the corner neighbors at least a C_4, and so is protected from becoming a defector. See the corner cells S, V, Y, and Z in Figure 3 or their equivalents in Figure 4.

Now we must answer as to why growth of some parts of the body, must result in a convex body in the next generation. Though corners can have values C_2, C_3, or C_4, the middle edge cooperators will always have value C_5, except in the singular case of a C_4, the straightedge cell adjacent to a 135°corner (Cell T in Figure 3 and its equivalent in Figure 4). We can show pictorially that any length of each straight and stepping edge will cause uniform lateral growth, and dominance or lack thereof at the corners will merely affect whether or not an intermediate edge will be introduced at that corner.

Thus, B_C will grow, and in the next generation it will also be a convex body containing at least a C_4, satisfying conditions for infinite expansion.

We have shown, in an 8-neighbor system where $C_4 > D_3$, a convex body of cooperators with at least a C_4 in an infinite sea of defectors will grow indefinitely as a convex body.

□

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