SPANNING FOREST GAMES

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Cooperative Game Theory REU

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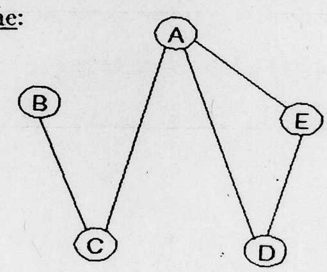
A *cooperative game* is a pair (N,w) where N is a set of *players* $\{1, 2, \ldots, n\}$, and w is a real-valued function on the subsets of N, where $w(\emptyset) = 0$. The subsets of N, denoted by S, are called *coalitions.* The number $w(S)$ represents the value the individual players in S can obtain by cooperating as a group. A cooperative game is *superadditive* if $w(S \cup T) \geq w(S) + w(T)$ for all coalitions S and T satisfying $S \cap T = \emptyset$.

A *graph* is an ordered pair $G=(V,E)$, where V is a nonempty finite set of *vertices,* and E is a set of pairs of elements of V called *edges*; we sometimes denote V and E as $V(G)$ and $E(G)$, respectively [1]. A *subgraph of G* is any graph H for which $V(H) \subseteq V(G)$, and $E(H) \subseteq E(G)$ [1]. *The subgraph of G on* $S \subseteq V(G)$ is (S,E') where $E'=\{ \{a,b\} \in E(G): a,b \in S\}$. A graph G is *connected* if for every pair of vertices there is a *path* between the two vertices; in other words, there exists a way to get from one vertex to the other without crossing any vertex more than once; a graph is *disconnected* otherwise [1]. A *component* of a graph is a subgraph which is itself connected [1]. The *connectivity of G* corresponds to the minimum number of vertices which, when removed, disconnects G [1].

A *cycle* is a closed path, and a *tree* is a connected graph which has no cycles [1]. For a connected graph G, a *spanning tree, T,* is a tree subgraph of G where $V(G) = V(T)$ [1]. A *forest* is a graph which has no cycles; hence, if G is disconnected, then a forest consists of the spanning trees of each of the graph's components [1]. We define a *spanning forest, F,* of G to be a forest subgraph of G where $V(G) = V(F)$ and no additional edge can be added without creating a cycle.

A *graph game* is a game defined on a graph. We define one such game, a *spanning forest game,* to be an ordered pair (G,w) where $G = (V,E)$ is a graph, and w is a real-valued function on the subsets of V defined as follows: $w(S)$ is the number of edges in a spanning forest of the subgraph on S. In other words, $w(S) = |S| - c$, where c is the number of components of S.

<u>Example of Spanning Forest Game:</u>



Notation: w({A,B,C}) = w(ABC)

w(A) = w(B) = w(C) = w(D) = w(E) = 0,

w(AB) = 0, w(AC) = 1, w(AD) = 1, w(AE) = 1, w(BC) = 1,

w(BD) = 0, w(BE) = 0, w(CD) = 0, w(CE) = 0, w(DE) = 1,

w(ABC) = 2, w(ABD) = 1, w(ABE) = 1, w(ACD) = 2, w(ACE) = 2,

w(ADE) = 2, w(BCD) = 1, w(BCE) = 1, w(BDE) = 1, w(CDE) = 1,

w(ABCD) = 3, w(ABCE) = 3, w(ABDE) = 2, w(ACDE) = 3, w(BCDE) = 2,

w(ABCDE) = 4