Nucleolus and Group Monotonicity

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Abstract

A coalition game (N, v) models a situation in which gains can be made by collaboration. $N = \{1,2,\ldots,n\}$ is the set of players and $v(S)$ is the real-number gain achievable for each nonempty subset S of players (called coalitions). We assume that $v(S) + v(T) \leq v(S \cup T)$ for each disjoint pair of players S and T. With this assumption, all players collaborating results in the largest combined payoff $v(N)$. Now the question is how to fairly allocate the gains obtained through collaboration, that is, what payoffs $x_1, x_2, \ldots, x_n$ should be given to the players subject to the constraint $\sum_{i \in N} x_i = v(N)$? The nucleolus $\mu(v)$ is an allocation method that is group rational: $\sum_{i \in S} \mu_i \geq v(S)$ for all coalitions S whenever such an allocation exists. Excess of some allocation x on some coalition S, or $e(x, S)$, is defined as $v(S) - \sum_{i \in S} x_i$, which specifies the "dissatisfaction" that the players in S have against x. The nucleolus is the allocation that successively minimizes the largest excesses. Unfortunately, the nucleolus is not always group monotone: an increase in one of the coalition gains $v(S)$ may result in a decrease in the nucleolus allocation $\mu(N, v)_i$ for some player $i \in S$. We try to characterize all 4-player coalition games at which the nucleolus is group monotone.

Examples

1. Super-additive game G_1:

s:	{ABCD}	{ABC}	{ABD}	{ACD}	{BCD}	{AB}	{AC}	{AD}	{BC}	{BD}	{CD}	{A}	{B}	{C}	{D}
v(S):	**120**	108	96	84	0	24	0	0	0	0	0	0	0	0	0

2. Nucleolus of G_1:

i:	A	B	C	D
$x_n(\mathbf{v}, \boldsymbol{i})$:	84	18	12	6

3. Excess and excess vector of nucleolus of G_1:

s:	{ABC}	{ABD}	{ACD}	{BCD}	{AB}	{AC}	{AD}	{BC}	{BD}	{CD}	{A}	{B}	{C}	{D}
$e(x_n, S)$:	-6	-12	-18	-36	-78	-96	-90	-30	-24	-18	-84	-18	-12	-6

$\theta(x_n)$:	-6	-6	-12	-12	-18	-18	-18	-24	-30	-36	-78	-84	-90	-96

4. "Important" collection of G_1:

$$\{\{\{A, B, C\}, \{D\}\}, \{\{A. B. D\}, \{C\}\}, \{\{A, C, D\}, \{B\}\}\}$$

5. System of equations obtained from the "important collection" and efficiency constraint:

$$e(x^*, \{A, B, C\}) = e(x^*, \{D\})$$
$$e(x^*, \{A, B, D\}) = e(x^*, \{C\})$$
$$e(x^*, \{A, C, D\}) = e(x^*, \{B\})$$
$$x_A^* + x_B^* + x_C^* + x_D^* = v(N)$$

6. Algebraic expression of the nucleolus of G_1:

$$x_A^* = \frac{1}{2}[v(ABC) + v(ABD) + v(ABD) - v(B) - v(C) - v(D) - v(ABCD)]$$
$$x_B^* = \frac{1}{2}[v(B) - v(ACD) + v(ABCD)]$$
$$x_C^* = \frac{1}{2}[v(C) - v(ABD) + v(ABCD)]$$
$$x_D^* = \frac{1}{2}[v(D) - v(ABC) + v(ABCD)]$$

Examples: Remark

In the *Example* section we illustrated the process of obtaining the algebraic expression of the nucleolus from a given game. Each game corresponds to some unique nucleolus and some "important" (balanced and relatively high in excess) collection. From the "important" collection, we can derive an algebraic expression for the nucleolus of the given game. From this expression, we can verify whether or not the nucleolus is group monotone or not at this game. Thus, if we can characterize all "important" collections for 4-player games, then we can categorize the classes of games that these collections correspond to, and categorize them by the group monotonicity of their nucleolus.

Balanced Collections

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{{{1}, {2, 3, 4}}, {{2}, {1, 3, 4}}, {{3}, {1, 2, 4}}, {{4}, {1, 2, 3}},
 {{1, 2}, {3, 4}}, {{1, 3}, {2, 4}}, {{1, 4}, {2, 3}}, {{1}, {2}, {3, 4}},
 {{1}, {3}, {2, 4}}, {{1}, {4}, {2, 3}}, {{2}, {3}, {1, 4}},
 {{2}, {4}, {1, 3}}, {{3}, {4}, {1, 2}}, {{1, 2}, {1, 3, 4}, {2, 3, 4}},
 {{1, 3}, {1, 2, 4}, {2, 3, 4}}, {{1, 4}, {1, 2, 3}, {2, 3, 4}},
 {{2, 3}, {1, 2, 4}, {1, 3, 4}}, {{2, 4}, {1, 2, 3}, {1, 3, 4}},
 {{3, 4}, {1, 2, 3}, {1, 2, 4}}, {{1}, {2}, {3}, {4}},
 {{1}, {2, 3}, {2, 4}, {3, 4}}, {{1}, {2, 3}, {2, 4}, {1, 3, 4}},
 {{1}, {2, 3}, {3, 4}, {1, 2, 4}}, {{1}, {2, 4}, {3, 4}, {1, 2, 3}},
 {{2}, {1, 3}, {1, 4}, {3, 4}}, {{2}, {1, 3}, {1, 4}, {2, 3, 4}},
 {{2}, {1, 3}, {3, 4}, {1, 2, 4}}, {{2}, {1, 4}, {3, 4}, {1, 2, 3}},
 {{3}, {1, 2}, {1, 4}, {2, 4}}, {{3}, {1, 2}, {1, 4}, {2, 3, 4}},
 {{3}, {1, 2}, {2, 4}, {1, 3, 4}}, {{3}, {1, 4}, {2, 4}, {1, 2, 3}},
 {{4}, {1, 2}, {1, 3}, {2, 3}}, {{4}, {1, 2}, {1, 3}, {2, 3, 4}},
 {{4}, {1, 2}, {2, 3}, {1, 3, 4}}, {{4}, {1, 3}, {2, 3}, {1, 2, 4}},
 {{1, 2}, {1, 3}, {1, 4}, {2, 3, 4}}, {{1, 2}, {2, 3}, {2, 4}, {1, 3, 4}},
 {{1, 3}, {2, 3}, {3, 4}, {1, 2, 4}}, {{1, 4}, {2, 4}, {3, 4}, {1, 2, 3}},
 {{1, 2, 3}, {1, 2, 4}, {1, 3, 4}, {2, 3, 4}}}
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Table 1. Minimally balanced collections for 4-player game.

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{{{1}, {2, 3, 4}}, {{1, 2}, {3, 4}},
 {{1}, {2}, {3, 4}}, {{1, 2}, {1, 3, 4}, {2, 3, 4}},
 {{1}, {2}, {3}, {4}}, {{1}, {2, 3}, {2, 4}, {3, 4}},
 {{1}, {2, 3}, {2, 4}, {1, 3, 4}}, {{1, 2}, {1, 3}, {1, 4}, {2, 3, 4}},
 {{1, 2, 3}, {1, 2, 4}, {1, 3, 4}, {2, 3, 4}}}
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Table 2. Non-isomorphic minimally balanced collections for 4-player game.

Theory

Definition 1.
A collection B of coalitions is said to be *balanced* if we can find some positive weight λ_S between 0 and 1 for each coalition S in the collection such that for each i in N, $\sum_{S \in B:\, i \in S} \lambda_S = 1$.

Definition 2.
Let (N, v) be a coalitional game, and let x be an efficient allocation. For every real value α, denote $D(\alpha, x)$ the collection of nonempty coalitions S satisfying $e(x, S) \geq \alpha$:
$D(\alpha, x) = \{S \subset N \mid S \neq \emptyset, e(x, S) \geq \alpha\}$.

Theorem 1. (Kohlberg, 1970)
A necessary and sufficient condition for imputation x to be the nucleolus of a super-additive game (N, v) is for $D(\alpha, x)$ to be a balanced collection for every real value α for which this collection is nonempty.

Theory: Remark

Theorem 1 illustrates that the nucleolus of each game (N, v) corresponds to some structure $\mathrm{K}(\mathrm{N}, v) = \{D(\alpha_1, x), D(\alpha_2, x), \ldots, D(\alpha_q, x)\}$, where $K_1 \cup K_2 \ldots \cup K_j$ is balanced for all $1 \leq j \leq q$.
From $\mathrm{K}(\mathrm{N}, v)$ we can extract the "important" collection $B(N, v)$ by including in B the first m non-singleton sets in K, such
$\sum_{i=1}^{m}(|B_i| - 1) \geq n - 1$ and $\sum_{i=1}^{m-1}(|B_i| - 1) < n - 1$.

Results

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{{{{1}, {2, 3, 4}}, {{2}, {3}, {1, 4}}}, {{{1}, {2, 3, 4}}, {{2}, {1, 3}, {1, 4}}}, {{{1}, {2, 3, 4}}, {{2, 3}, {2, 4}, {1, 3, 4}}},
 {{{1}, {2, 3, 4}}, {{2, 3}, {1, 2, 4}, {1, 3, 4}}}, {{{1}, {2, 3, 4}}, {{2}, {3, 4}}, {{3}, {1, 4}}}, {{{1}, {2, 3, 4}}, {{2}, {3, 4}}, {{3}, {1, 2, 4}}},
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 {{{1}, {2, 3, 4}}, {{2}, {1, 3, 4}}, {{2, 3}, {1, 2, 4}}}, {{{1}, {2, 3, 4}}, {{1, 2}, {3, 4}}, {{3}, {1, 4}}},
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 {{{1, 2}, {1, 3, 4}, {2, 3, 4}}, {{1, 3}, {2, 4}}}, {{{1, 2}, {1, 3, 4}, {2, 3, 4}}, {{1, 3}, {1, 2, 4}}}, {{{1}, {2, 3}, {2, 4}, {1, 3, 4}}}}
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Result 1. 43 non-isomorphic, non-group monotonic "important" collections

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{{{{1}, {2, 3, 4}}, {{2}, {3}, {4}}}, {{{1}, {2, 3, 4}}, {{1, 2}, {1, 3}, {1, 4}}}, {{{1}, {2, 3, 4}}, {{2, 3}, {2, 4}, {3, 4}}},
 {{{1}, {2, 3, 4}}, {{1, 2, 3}, {1, 2, 4}, {1, 3, 4}}}, {{{1}, {2, 3, 4}}, {{2}, {3, 4}}, {{3}, {4}}}, {{{1}, {2, 3, 4}}, {{2}, {3, 4}}, {{3}, {2, 4}}},
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 {{{1}, {2, 3, 4}}, {{1, 2}, {1, 3, 4}}, {{3}, {4}}}, {{{1}, {2, 3, 4}}, {{1, 2}, {1, 3, 4}}, {{3}, {2, 4}}},
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 {{{1, 2}, {3, 4}}, {{1}, {2}}, {{3}, {4}}}, {{{1, 2}, {3, 4}}, {{1}, {2}}, {{3}, {1, 4}}}, {{{1, 2}, {3, 4}}, {{1}, {2}}, {{3}, {1, 2, 4}}},
 {{{1, 2}, {3, 4}}, {{1}, {2}}, {{1, 3}, {1, 4}}}, {{{1, 2}, {3, 4}}, {{1}, {2}}, {{1, 3}, {2, 4}}}, {{{1, 2}, {3, 4}}, {{1}, {2}}, {{1, 3}, {1, 2, 4}}},
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 {{{1, 2}, {3, 4}}, {{1, 3}, {2, 4}}, {{1, 4}, {2, 3}}}, {{{1, 2}, {3, 4}}, {{1, 2, 3}, {1, 2, 4}}, {{1}, {2}}},
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 {{{1}, {2}, {3, 4}}, {{3}, {1, 4}}}, {{{1}, {2}, {3, 4}}, {{1, 3}, {1, 4}}}, {{{1}, {2}, {3, 4}}, {{1, 3}, {2, 4}}},
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 {{{1, 2}, {1, 3, 4}, {2, 3, 4}}, {{1, 3}, {1, 4}}}, {{{1, 2}, {1, 3, 4}, {2, 3, 4}}, {{1, 2, 3}, {1, 2, 4}}}, {{{1}, {2}, {3}, {4}}},
 {{{1}, {2, 3}, {2, 4}, {3, 4}}}, {{{1, 2}, {1, 3}, {1, 4}, {2, 3, 4}}}, {{{1, 2, 3}, {1, 2, 4}, {1, 3, 4}, {2, 3, 4}}}}
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Result 2. 62 non-isomorphic, group monotonic "important" collections

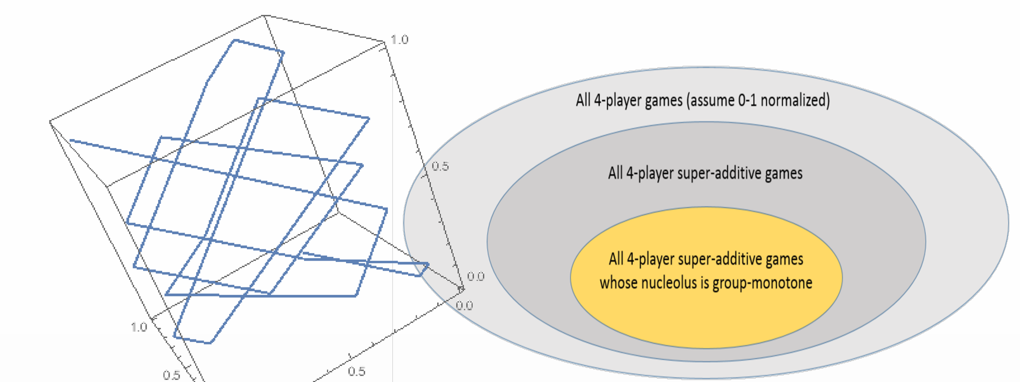
$$\begin{pmatrix}
1 & \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -\frac{1}{2} & -\frac{1}{2} & 0 & 1 \\
0 & 0 & \frac{1}{2} & 0 & 1 & 0 & 0 & 0 & 0 & 0 & -1 & -\frac{1}{2} & 0 & 0 & \frac{1}{2} \\
0 & \frac{1}{2} & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & -1 & 0 & -\frac{1}{2} & 0 & \frac{1}{2} \\
0 & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & 0 & 0 & 1 & 0 & 0 & 0 & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & 0 & \frac{1}{2} \\
0 & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & 0 & 0 & 0 & 1 & 0 & 0 & -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & 0 & -\frac{1}{2} \\
0 & -\frac{1}{2} & 0 & -1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & \frac{1}{2} & 0 & -\frac{1}{2} \\
0 & 0 & -\frac{1}{2} & -1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & \frac{1}{2} & 0 & 0 & -\frac{1}{2} \\
0 & -\frac{1}{2} & -\frac{1}{2} & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} & 1 & -1 \\
1 & \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{2} & -1 & -\frac{1}{2} & 0 & 1 \\
0 & 0 & 0 & \frac{1}{2} & 1 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{2} & -1 & 0 & 0 & \frac{1}{2} \\
0 & \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & 0 & 1 & 0 & 0 & 0 & 0 & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & 0 & \frac{1}{2} \\
0 & \frac{1}{2} & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & -1 & -\frac{1}{2} & 0 & \frac{1}{2} \\
0 & -\frac{1}{2} & -1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & \frac{1}{2} & 0 & -\frac{1}{2} \\
0 & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & 0 & 0 & 0 & 0 & 1 & 0 & \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & 0 & -\frac{1}{2} \\
0 & 0 & -1 & -\frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 1 & \frac{1}{2} & 0 & 0 & 0 & -\frac{1}{2} \\
0 & -\frac{1}{2} & -1 & -\frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} & 1 & -1 \\
1 & 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{2} & -\frac{1}{2} & -1 & 0 & 1 \\
0 & -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & 1 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & 0 & \frac{1}{2} \\
0 & 0 & 0 & \frac{1}{2} & 0 & 1 & 0 & 0 & 0 & 0 & -\frac{1}{2} & 0 & -1 & 0 & \frac{1}{2} \\
0 & 0 & \frac{1}{2} & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & -\frac{1}{2} & -1 & 0 & \frac{1}{2} \\
0 & -1 & -\frac{1}{2} & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & \frac{1}{2} & 0 & 0 & -\frac{1}{2} \\
0 & -1 & 0 & -\frac{1}{2} & 0 & 0 & 0 & 0 & 1 & 0 & \frac{1}{2} & 0 & 0 & 0 & -\frac{1}{2} \\
0 & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 1 & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & 0 & -\frac{1}{2} \\
0 & -1 & -\frac{1}{2} & -\frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 & 1 & -1
\end{pmatrix}
\cdot
\begin{pmatrix}
v[\{1\}] \\ v[\{2\}] \\ v[\{3\}] \\ v[\{4\}] \\ v[\{1,2\}] \\ v[\{1,3\}] \\ v[\{1,4\}] \\ v[\{2,3\}] \\ v[\{2,4\}] \\ v[\{3,4\}] \\ v[\{1,2,3\}] \\ v[\{1,2,4\}] \\ v[\{1,3,4\}] \\ v[\{2,3,4\}] \\ v[\{1,2,3,4\}]
\end{pmatrix}
\geq
\begin{pmatrix}
0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0
\end{pmatrix}$$

Result 3. Corresponding polytope in 15-D for nucleolus of G_1

Results: Remark

Out of 105 non-isomorphic nucleoli for 4-player games, 43 are non-group monotonic and 62 are group monotonic. Each of the 105 nucleoli corresponds to a whole class of isomorphic nucleoli and their corresponding regions in 15-D game space, as illustrated in Result 3. No known method provides exact volume for these polytopes.

Statistical Analysis



http://mathematica.stackexchange.com/questions/111892/particles-bouncing-in-a-3d-box

50,000 games in the 4-player super-additive game space were randomly generated with hit-and-run algorithm. 29,523 games had group monotonic nucleolus, and 20,477 games had non-group monotonic nucleolus; 40.954% of games generated were had non-group monotonic nucleolus.

Conjecture

$\frac{43}{43+62} = 40.9524\%$, which is a coincidental close agreement with the statistical result. Perhaps each of the isomorphic nucleolus whose non-isomorphic form represented in Result 1 and Result 2 corresponds to polytopes in 15-D whose total volume divides up the 4-player super additive game space into 105 equal-volume regions. Further research is needed to verify this claim.

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