

Maximal Flow Problems and Cooperative Games

A Senior Comprehensive Project

by

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I hereby recognize and pledge to fulfill my responsibilities, as defined in the Honor Code, and to maintain the integrity of both myself and the College community as a whole.

Pledge:

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Abstract

This project involves looking at maximal flow problems and cooperative games. First, a maximal flow problem involves a graph that has flow moving along its arcs from a beginning node to an end node. A cooperative game involves a set of players who can combine their assets to form coalitions. Each of the coalitions has a value, the worth of that coalition, and there are methods to determine how to distribute this among the players. Combining these two concepts, we can arrive at a max flow game where the arcs of the graph are players and their worth is the maximal flow through those arcs. Then, we can look at various methods of distribution to the players in these games.

1 Chapter One: Maximal Flow Problems

A *graph* is a set of junction points (nodes) with lines or branches connecting certain pairs of points, and a *network* is a graph with a flow of some type in its branches. The upper limit to the feasible amount of flow in a branch in a specified direction is the *flow capacity* of the branch in that direction. These definitions provide the background for the maximal flow problem. The maximal flow problem involves a connected network having a single source (a node where the flow moves away from it) and a single sink (a node where the flow moves towards it). At all other nodes what flows in must also flow out. The rate (or total quantity) of flow along branch (i, j) from node i to node j can be any nonnegative quantity not exceeding the specified flow capacity c_{ij} , and the objective is to determine the feasible pattern of flows through the network that maximizes the total flow from the source to the sink. In other words, the maximal flow problem seeks to find the largest amount that can be moved from the source to the sink when there are limits on how much can be moved along each branch.

The following is an example of a maximal flow problem where the node designated by s is the source and the node designated by t is the sink:



