Some observations on a notion of consistency

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0 Introduction

Reduced game concepts model renegotiation, within subsets of the players, of the payoffs given by a particular value. A value is consistent when any reallocation awards players exactly their payoff in the original game. One of our eventual goals is to find conditions on a reduced game concept which would guarantee the existence of a value consistent with respect to the reduced game concept. Here we explore some techniques possibly useful in that endeavor.

This paper arose out of an earlier, erroneous paper of mine, which claimed to prove an applicable result. This paper makes no such claim; first, I have here left a key conjecture unproven (though I believe it is true); more importantly, one of the assumptions is so strict that probably any theorem is true only vacuously. The purpose of this paper is just to "see how far we can really get" within the framework of my ideas over the summer. Thanks to David Housman for finding the error. Apologies to all whom I misled.

1 Basic Notions

A game is a pair (N, v) where $N = \{1, 2, \cdots, n\}$ represents the players and $v: 2^N \to \mathbb{R}$ with $v(\emptyset) = 0$ represents the worths of coalitions of players. A value φ associates with each (N, v) a vector $\varphi(N, v) \in \mathbb{R}^n$ where $\varphi_i(N, v)$ is the payoff to player i.

Example: The Shapley value Sh is defined by

$$Sh_i(N, v) = \sum_{S \subseteq N} \frac{(n-s)!(s-1)!}{n!}\left[v(S) - v(S \backslash i)\right]$$

A reduced game concept RG associates with each 4-tuple (N, v, S, φ), where $S \subseteq N$, a new game $RG(N, v, S, \varphi)$ on player set S. Since we will encounter no serious ambiguity regarding RG and N, this new game will be denoted by either (S, v_S^{φ}) or (S, v_S), usually the latter.

Examples: The Hart/Mas-Colell reduced game concept is defined by

$$v_S(T) = v(T \cup (N \backslash S)) - x(N \backslash S) \text{ where } x = \varphi\left(T \cup (N \backslash S), v|_{2^{T \cup (N \backslash S)}}\right)$$

The Davis/Maschler reduced game concept is defined by

$v_S(S) = x(S)$, where $x = \varphi(N, v)$

$v_S(T) = \max_{R \subseteq N \setminus S} (v(T \cup R) - x(R))$ for $T \subset S$

2 Definitions

Notation: For $x \in \mathbb{R}^n$, $x(S) := \sum_{i \in S} x_i$

Identify $\{i\}$ and i.

Write I_k for $\{1, 2, \cdots, k\}$.

Definitions: A game (N, v) is

0-monotonic if $\forall i \in N, \forall S \subseteq N \setminus i$, we have $v(S \cup i) \geq v(S) + v(i)$

superadditive if $\forall S, T \subseteq N$ with $S \cap T = \emptyset$, we have $v(S \cup T) \geq v(S) + v(T)$

convex if $\forall i \in N, \forall S \subseteq T \subseteq N \setminus i$, we have $v(S \cup i) - v(S) \leq v(T \cup i) - v(T)$

or, equivalently, if $\forall S, T \subseteq N$, we have $v(S) + v(T) \leq v(S \cup T) + v(S \cap T)$

Definition: We say that $i \in N$ is a *dummy* in (N, v) if $\forall S \subseteq N, v(S \cup i) = v(S)$

Definitions: A value φ satisfies

DPP (dummy player property) if $\varphi_i(N, v) = 0$ whenever i is a dummy in (N, v)

SYM (symmetry) if for any permutation τ on N we have $\tau(\varphi(v)) = \varphi(\tau(v))$ where for $x \in \mathbb{R}^n$,

$\tau(x)_i := x_{\tau(i)}$, and for $S \subseteq N, \tau(v)(S) := v(\tau(S))$

COV (covariance under strategic equivalence) if

$\alpha\{> 0, \beta \in \mathbb{R}^n$ so that $\forall S \subseteq N$: $w(S) = \alpha v(S) + \beta(S)\} \Rightarrow \varphi(N, w) = \alpha\varphi(N, v) + \beta$. If the bracketed condition holds, then v and w are said to be *strategically equivalent.*

IR (individual rationality) at (N, v) if $\forall i \in N$, we have $\varphi_i(N, v) \geq v(i)$

EFF (efficiency) if $\sum_{i \in N} \varphi_i(N, v) = v(N)$