A Characterization of the Extreme Monotonic Extensions of a Partially Defined Game

Roger Lee

September 29, 1992

Drew University REU

Prof. David Housman

When only limited information exists about the worths of certain subsets of individuals in a game, standard methods cannot compute payoffs. One solution is to allocate as dictated by some specific extension of the game. The extreme points of monotonic extensions are characterized.

1 Background

A *cooperative game* is a pair (N, w) where $N = \{1, 2, \cdots, n\}$ is a set of *players* and $w: 2^N \to \mathbb{R}$ with $w(\emptyset) = 0$ gives the worth obtainable by the cooperation of each subset of players. A *value* associates with each (N, w) a vector in $\mathbb{R}^n$ representing the payoff to each player. A game is *monotonic* if $w(S) \leq w(T)$ whenever $S \subseteq T$.

Letscher (1990) defines a *partially defined cooperative game (PDG)* to be a triple (N, Ω, v) where $\Omega \subseteq 2^N$ are the coalitions whose worths are known and $v: \Omega \to \mathbb{R}$ gives these worths. We require $\emptyset, N \in \Omega$. An *extension* of this PDG is a game (N, w) with $v(S) = w(S)$ for all $S \in \Omega$. We define a *partial extension* of this PDG to be a PDG $(N, \bar{\Omega}, \bar{v})$ where $\Omega \subseteq \bar{\Omega}$ and $v(S) = \bar{v}(S)$ for all $S \in \Omega$. Define this PDG to be *monotonic* if $v(S) \leq v(T)$ for all $S, T \in \Omega$ with $S \subseteq T$.

The object of Ventrudo and Wallman (1991) is to determine value on all PDGs $P = (N, \Omega, v)$. One approach is to find the set $\mathfrak{M}(P)$ of all monotonic extensions of the game, select some "central" point in this set, and apply some value to this game. A geometric characterization of $\mathfrak{M}(P)$ facilitates the selection of such a point.

View a game (N, w) as a vector in $\mathbb{R}^{2^n}$. It is easy to verify that $\mathfrak{M}(P)$ is a bounded convex set. An *extreme point* of a convex set C is an $x \in C$ such that if $c_1 + c_2 = 2x$ for some $c_1, c_2 \in C$, then $c_1 = c_2 = x$. We characterize ex($\mathfrak{M}(P)$), the extreme points of $\mathfrak{M}(P)$.

2 A Condition Sufficient for Extremity

Index each factor in the product $\{0,1\}^b$, where $b = 2^n - |\Omega|$, by a different element of $2^N \backslash \Omega$. For any $\alpha \in \{0,1\}^b$, any monotonic PDG $P = (N, \Omega, v)$, define the game (N, v^α) as follows. Arrange the elements of $2^N \backslash \Omega$ in order of nondecreasing cardinality: $S_0, S_1, \cdots, S_b$. Define $v^\alpha(S) = v(S)$ for all $S \in \Omega$. Assume that $v^\alpha(S_i)$ has been defined for all $i < l$. Define

$$v^\alpha(S_l) = \begin{cases} \max\{v^\alpha(S) \mid S \subseteq S_l \text{ and } (S \in \Omega \text{ or } S = S_i \text{ for some } i < l)\} & \text{if } \alpha(S_l) = 0 \\ \min\{v^\alpha(S) \mid S \supseteq S_l,\ S \in \Omega\} & \text{if } \alpha(S_l) = 1 \end{cases}$$

Theorem: If $(N, v^\alpha) = (N, w)$ for some $\alpha \in \{0,1\}^b$, then $(N, w) \in \mathrm{ex}(\mathfrak{M}(P))$.

Proof: Ventrudo and Wallman (1991). □

We claim the converse fails. Define P by $N = \{12345\}$, $\Omega = \{S \subseteq N \mid |S| = 0, 1, 4, \text{ or } 5\}$,

$$v(12345) = 2,\ v(i) = 0,$$

$$v(2345) = 1, \text{ otherwise } v(ijkl) = 2.$$

Define the extension (N, w) by $w(123) = w(234) = w(235) = w(23) = w(12) = 1;$

$$w(124) = w(125) = 2; \text{ for other } |S| = 2 \text{ or } 3,\ w(S) = 0.$$

which is monotonic. To show w extreme, suppose $\exists \Delta \in \mathbb{R}^{2^n}$ such that $w \pm \Delta$ are monotonic. Then

$$w(23) = w(234) = w(235) = v(2345) \Rightarrow \Delta(23) = \Delta(231) = \Delta(235) = 0$$

$$w(123) = w(23) \Rightarrow \Delta(123) = 0$$

$$w(12) = w(123) \Rightarrow \Delta(12) = 0$$

Clearly for all other $S \subseteq N$, $\Delta(S) = 0$, making Δ the zero vector. So w is extreme.

However there is no α such that $v^\alpha = w$, because $v^\alpha(12)$ can only be 0 or 2, never 1.