FOREST-SIZE GAMES

by

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I. Introduction and Notation

A Cooperative Game is an ordered pair (N, w), where $N = \{1, 2, \dots, n\}$ represents players in a game, and w is a real-valued function, called the worth function, which maps subsets of N onto real numbers, with the condition that $w(\emptyset) = 0$. A subset of N, denoted S, is called a coalition; when $S = N$, we refer to it as the grand coalition. We will be looking at a cooperative game with the property of superadditivity; a game is superadditive if $\forall\, S,T \subseteq N : S \bigcap T = \emptyset,\ \ w(S \bigcup T) \geq w(S) + w(T)$.

An allocation for a cooperative game is a vector $x = (x_1, x_2, \dots, x_n)$, where x_i represents an individual allocation, or payoff, to player i. An allocation method is a function which maps cooperative games to an allocation. The allocation method which will be used in this study will have the following four properties:

(1) Efficiency: $\sum_{i=1}^{n} x_i = w(N)$.

(2) Equal Treatment: If $w(S \bigcup \{i\}) = w(S \bigcup \{j\})$ for all $S \subseteq N - \{i, j\}$, then $x_i(N, w) = x_j(N, w)$.

(3) No Free Lunch: Let $i \in N$. If $w(S \bigcup \{i\}) = w(S) + w(i)$ for all S s.t. $i \notin S$, then $x_i(N, w) = w(i)$.

(4) Additivity: $x_i(N, v+u) = x_i(N, v) + x_i(N, u)$ for all games (N, v) and (N, u).

There is a unique allocation method which satisfies the preceding four properties; it is called the Shapley value. A formula for finding the Shapley value of a cooperative game is given:

$$\phi_i(N, w) = \sum_{S \subseteq N} \frac{(s-1)!\,(n-s)!}{n!}\,[w(S) - w(S - \{i\})], \text{ where } s = |S|.$$

This project will look at the Shapley value to a specially-defined game called a forest-size game. A forest-size game is represented as a graph, $G=(V, E)$ with n vertices. $V(G)$ represents the set of vertices in the graph G, and $E(G)$ represents the set of symmetric pairs of an irreflexive, symmetric relation on V. The following terms will be used throughout the report:

(1) Adjacency: Two vertices u, v are set to be adjacent, if the edge uv $\in E(G)$. Similarly, if two edges, uv and uw are distinct edges of a graph G, then uv and uw are adjacent.

(2) Incidence: If an edge uv $\in E(G)$, then vertices u and v are said to be incident with edge uv.

(3) Degree: The degree of a vertex v is the number of edges incident with v.