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The Shapley Value and Partially Defined Games

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Introduction

Many mathematicians are realizing the usefulness of cooperative game theory in the process of allocating costs or benefits among the participants of joint endeavors. Realizing that in reality, the determination of all coalitional worths may be prohibitively expensive or impractical, this paper is aimed at providing possible allocation methods for games which have unknown coalitional worths.

Definitions

In this paper, all games in which some coalitional worths are not known are referred to as partially defined games, or PDG's. Each game consists of a set of players, N, with $N = \{1, 2, ..., n\}$. Let M be a subset of N such that $1, n \in M$. A partially defined game with respect to M, or an *M-game*, is a real valued function ω on $\{S \subseteq N: |S| \in M\}$. The real number $\omega(S)$ is often called the worth of the coalition S. By defining M so that it always contains 1 and n, we insure that the worth of the singleton and grand coalitions are known, in addition to the worths of any other coalition whose size is in the set M.

An allocation for the M-game is a vector of payoffs $x \in \mathbb{R}^n$. An allocation method is a function from a class of games to allocations $\mathbb{R}^n$ which attempts to fairly distribute the costs or benefits of the joint venture. The

Shapley value φ is an allocation on N-games defined by the formula

$$\varphi_i(\omega) = \sum_{S\subseteq N,\, i\in S} \frac{(s-1)!\,(n-s)!}{n!}\,[\omega(S) - \omega(S-\{i\})]$$

where s=|S| and n=|N|. This formula tells us that the Shapley value to player i in game w is player i's average marginal contribution over all possible orderings of players. An equivalent formula obtained through algebraic manipulation is

$$\varphi_i(\omega) = \frac{1}{n}\sum_{|S|\in N} \left[\sum_{|S|=s,\, i\in S} \binom{n-1}{s-1}^{-1} \omega(S) - \sum_{|S|=s,\, i\notin S} \binom{n-1}{s}^{-1} \omega(S)\right]$$

This form implies that player i should obtain an average over coalition sizes of the average worth of coalitions containing i minus the average worth of those not containing i. This becomes relevant when studying certain classes of M-games.

When dealing with a PDG, the unknown worths of coalitions can be estimated based on the class of game being considered. If Ω is a class of N-games, then the N-game $\hat{\omega}$ is called a Ω-extension of the M-game ω if $\hat{\omega} \in \Omega$ and $\hat{\omega}(S)=\omega(S)$ for all $|S| \in M$. In this paper, the set of all Ω-extensions of ω will be denoted ext(ω). Once the set of all Ω-extensions has been characterized it becomes possible to find a central Ω-extension of the M-game ω. Assuming a uniform probability distribution of the extensions, this paper examines the use of two different central extensions. First, the geometric center, denoted by centroid ext(ω) = $\mathbf{C}$, is used as a reasonable estimate of the underlying N-game. Second, the coordinate center of the ext(ω) is used as another means of approximating the central extension of the M-game. The central N-game, defined by either the centroid or the coordinate center, then becomes a sensible extension on which to apply the Shapley value.