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Discoveries: Final Paper

The Flaw in the Machiavelli Index

The definition of a proper simple game was taken from an earlier paper written on this subject by the author in 2013.

A ***proper simple game*** $G = (N, W)$ is a set of players, N, and a set of winning coalitions, W, that satisfies:

1. Winning coalitions are sets of players (if $S \in W$, then $S \subseteq N$),

2. Unanimity wins ($N \in W$),

3. No player is detrimental (if $A \subseteq B$ and $A \in W$, then $B \in W$), and

4. Only one coalition can be winning at any particular time (if $A \in W$, then the complement $N \setminus A \notin W$).

A ***consistent game*** satisfies the additional property:

5. For each pair of players, A and B, if for some set of players $S_0 \subseteq N \setminus \{A, B\}$, $A \cup S_0 \in W$ and $B \cup S_0 \notin W$, then for all $S \subseteq N \setminus \{A, B\}$, $B \cup S \in W$ implies that $A \cup S \in W$.

A ***weighted voting game*** is a proper simple game (N, W) such that

$W = [q; w_1, w_2, \ldots, w_n]$, where q is the number of votes needed to win, and w_i is the number of votes possessed by some player i.

Lemma 1: All weighted voting games satisfy the consistency criterion.
Proof by Contradiction:
Suppose there existed a game W that was not consistent. Then, there exist two players, A and B, such that
$A \cup S_0 \in W$, $B \cup S_0 \notin W$, $B \cup S_1 \in W$, and $A \cup S_1 \in \notin W$.
$A \cup S_0 \in W$ implies $w_A + w_{S_0} \geq q$
$B \cup S_0 \notin W$ implies $w_B + w_{S_0} < q$
$B \cup S_1 \in W$ implies $w_B + w_{S_1} \geq q$
$A \cup S_1 \in \notin W$ implies $w_A + w_{S_1} < q$
But adding the first and third rows gives
$w_A + w_{S_0} + w_B + w_{S_1} \geq 2\,q$
and adding the second and fourth rows gives
$w_B + w_{S_0} + w_A + w_{S_1} < 2\,q$,
A contradiction.

The ***Machiavelli Index,*** ω(G), measures the power of players in a weighted voting game such that $\sum_{i=1}^{n} w_i = 1$. The Machiavelli Index is defined recursively. The Machiavelli Index defines the single player game, $G = (\{a\}, \{\{a\}\})$, $\omega(G) = \{1\}$.

A coalition, C, is a group of two or more players, with $C \subset N$. Before the official vote, the Machiavelli Index assumes that members of the coalition use a proper simple game to decide how they will vote. Each member of the coalition will vote as the group has decided in the official vote in front of the larger body. Thus, each players' power is the product of their intra-coalition power and the coalition's power within the larger body. Define size(C) to be the number of players in C.

The ***set of possible Machiavelli divisions*** for n players are all the z possible ways that n players could divide power in a weighted voting game such that $0 < w_i \leq 1$ for all players i. Let

$$\delta_n = \begin{pmatrix} w_{1,1} & w_{2,1} & \cdots & w_{n,1} \\ w_{1,2} & w_{2,2} & \cdots & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ w_{1,z} & \cdots & \cdots & w_{n,z} \end{pmatrix}$$

. In an n-player game, δ_k is presumed to be known for all $1 \leq k < n$.

For a given game $G = (N, W)$, the Machiavelli Index computes all possible partitions of players $P(N) = \{P_1, P_2, ..., P_s\}$ where P_s is a unique partition of N such that there exists at least one coalition $C_{s,1} \subset N$. Let $C_{s,t}$ refer to the t coalition in the s partition. Let group(P_s) be the number of coalitions in P_s. For each coalition $C_{s,t} \subseteq P_s \subseteq P(N)$, there are $z_{s,t} = \delta_{\text{size}(C_{s,t})}$ ways to divide power. For each P_S, there are $z_s = \prod_{t=1}^{\text{group}(s)} z_{s,t}$ possible combinations of weighted voting games that coalitions could use to determine how they vote before voting in front of the whole group. We will only choose ways to divide power if they treat identical players equally. For each way, a vertex is created in a directed graph F with each player's power under that system of coalitions. This is possible because all smaller games are known and at least one coalition exists. This system of coalition(s) and set of Machiavelli divisions gives an effective new proper simple game with minimal winning coalitions, that is smaller than the original game, and thus known via the inductive hypothesis. Furthermore, each player in a coalition is bolded at each vertex. Let $B_{s,t}$ be the set of bolded players at a given vertex $V_{s,t}$. An edge is drawn from one vertex u to another vertex v when $\omega_u(b) < \omega_v(b)$ for all bolded players $b \in B$ on the vertex v. Then the vertex v is said to dominate the vertex u. Given a game $G = (N, W)$, we construct the directed graph $F = (V, E)$,

In F, a vertex v is considered to be stable if it has no edges directed to stable verticies. Any verticies that have an edge going to a stable vertex are considered unstable. A players power in $G(N, W)$ is the mean of their power in each stable vertex.

This paper hoped to show that, for games with the consistency property,
1.) All stable verticies contain minimal winning coalitions.
2.) A unique stable/unstable labelling system exists for all weighted voting games.
3.) The Machiavelli Index has the monotonicity property.

Sadly, it was discovered that the Machiavelli Index does not satisfy #2 and is thus undefined, even if we restrict ourselves only to consistent games.

Proof

Let $G' = [\{1, 2, 3, 4, 5, 6\}, \{1234, 1235, 1236, 1245, 1246, 1256\}]$, where 1 and 2 are identical players and 3-6 are identical players. In other words, to win, both 1 and 2 and any two of 3-6 are needed. This game is consistent, because we could assign the 1 and 2 three votes each and the 3-6 one vote each, requiring a total of 8 votes to pass a bill.

Since lots of players look the same, there are functionally only 11 possible combinations of coalitions. I will refer to 1 and 2 as B's for big, and 3-6's as S for small. We could have:

BB
BBS
BBSS
BBSSS
BS
BSS
BSSS
BSSSS
SS
SSS
SSSS

No Stable Structures Exist.

A BB coalition is pointless, because both voters are needed to pass any bill anyway. It does not change the minimal winning coalitions.
A BBS coalition is not enough to win on its own and is vulnerable to the other 3 S's forming a coalition and only voting yes in the larger assembly if all three vote yes in the coalition. This would give these 3 S's veto power. Since the 2 B's can get veto power on their own, a BBS coalition cannot force anything better than $\frac{1}{5}$ of the power for the two B's.
A BBSS coalition is minimally winning. Per a 2013 paper I wrote on this subject, the only ways to divide power that treat the two B's equally are to give the B's $\frac{1}{3}$ and the S's $\frac{1}{6}$ or to give all players $\frac{1}{4}$. We will return to these possibilities in a moment.
A BBSSS coalition is more than minimally winning. Since it must treat the three S's equally, none of the three S's could be getting more than $\frac{1}{5}$ of the power. If they're not getting $\frac{1}{5}$ of the power, they could force $\frac{1}{5}$ of the power by forming an SSS coalition and forcing a unanimity vote between BBSSS.
A BS coalition is subject to the other three S's forcing a unanimity vote between BBSSS with a SSS coalition.
A BSS coalition means the only way to pass anything is to get the coalition's approval and the approval of the other B. These have the same minimal winning coalitions as the BBSS coalition and have the same possible ways to divide power.
A BSSS coalition still needs the other remaining B and so is similiar to a BBSSS coalition.
A BSSSS coalition needs the other B to pass anything, and thus cannot give any S more than $\frac{1}{6}$ of the power. Three of the S's could form an SSS coalition on their own and get $\frac{1}{5}$.
An SS coalition is prone to being completely ignored by a winning BBSS with the other two S's.
An SSS coalition can force a unanimity game between the two B's and the three S's if the three S's only approval a proposal if they all do. This gives each of the five $\frac{1}{5}$.

An SSSS coaltion still needs both B’s to pass anything and so cannot give more than $\frac{1}{6}$ to any S. Three of the S’s could ditch the other one and get $\frac{1}{5}$ on their own.

By these arguments, the only possible structures that might be undominated are a SSS coaltition that forces $\frac{1}{5}$
power for the 2 B’s and 3 S’s, a BBSS coalition that gives everyone $\frac{1}{4}$, and a BBSS coalition that gives $\frac{1}{3}$ to the B’s and $\frac{1}{6}$ to the S’s. The problem is that if we have the $\frac{1}{5}$ situation, a BBSS coalition with $\frac{1}{4}$ for its members would clearly be better for all of its members. But then, the two B’s could go to the two S’s that were not included in this coalition and offer them $\frac{1}{6}$, while keeping $\frac{1}{3}$ for themselves. Since the other two S’s had 0 power before, they will do this deal. But then, all four of the S’s are getting less than $\frac{1}{5}$ of the power and could force $\frac{1}{5}$ for three of them with the SSS coalition. Thus no structures are stable and the Machiavelli Index is undefined for this game.

References

Schrock, Peter. The Machiavelli Index. 8/2013.