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The Shapley Value for Partition Function Form Games

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Partition function form games were first introduced in Lucas and Thrall in 1963 as a generalized form of characteristic function form games. Both R.B. Myerson and E.M. Bolger have defined values on partition function form games. In this paper an extension of the Shapley value is sought which will satisfy linearity, efficiency, symmetry and dummy. Different axioms are then incorporated to place bounds on the various remaining parameters.

The following background information, along with 12 axioms and 7 definitions, is key to the paper.

Background Information:

$N = \{1,2,\ldots,n\}$ is the set of players in a n-person game.

$CL = \{S \mid S\subseteq N, S\neq\emptyset\}$ is the set of coalitions of N.

$PT(S) = \{\{S^1,\ldots,S^m\} \mid S^1 \cup \ldots \cup S^m = S, \forall j\ S^j \neq\emptyset, \forall k\ S^k\cap S^j=\emptyset$ if $k\neq j\}$ is the set of partitions of S.

$ECL = \{(S,P) \mid S\in CL, P\in PT(N-S)\}$ is the set of embedded coalitions, that is, the coalition S is faced with the players in N-S grouped by the partition P.

An n-player game in partition function form is any $W \in R^{ECL}$, that is, W is a function from embedded coalitions to real numbers. We may interpret W(S;P) to be the amount S would receive if the players in N-S cooperated according to the partition P.

A value is a function Φ from some class of n-player games to R^n, the set of allocations. That is, $\Phi_i(W)$ is the allocation to player i in the game W.

A game is <u>superadditive</u> if any combination of two coalitions has a higher value than the two coalitions alone: $W(S;P \cup \{T\}) + W(T;P \cup \{S\}) \leq W(S \cup T;P)$ for all disjoint coalitions S and T and partitions $\{S,T\} \cup P$ of N. A game is <u>coalition monotonic</u> if adding players to a coalition, without moving other players, raises its worth: $W(S;P) \leq W(T; \{R-T: R \in P\})$, for all $S \subseteq T$. A game is <u>partition monotonic</u> if a coalition's worth is higher the less cooperation there is among the players outside the coalition: $W(S;P) \leq W(S;Q)$ whenever Q is a refinement of P, i.e., $R \in Q \Rightarrow \exists S \in P$ such that $R \subseteq S$.

We also define a partial order on embedded coalitions in the following manner: $(T;Q) \geq (S;P)$ if $S \subseteq T$ and Q is a refinement of $\{R-S: R \in P\}$. A <u>unanimity game</u> for $(S;P)$ is the game W defined by $W(T;Q) = 1$ if $(T;Q) \geq (S;P)$ and $W(T;Q) = 0$ otherwise.

<u>Axioms and Definitions</u>:

For the following axioms and definitions, let $S,T \subseteq N$, $P \in PT(N-S)$ and $Q \in PT(N-T)$, and W and V be games on N. The definitions are given as conditions which hold for all games in some class G. The reference to the class G has been omitted from each definition.

<u>Definition 1</u>: Suppose $\pi:N \rightarrow N$ is any permutation of the set of players. Then π acts as a permutation on CL and ECL in the following way:

$\pi(S) = \{\pi(j) \mid j \in S\}$, $\forall S \in CL$, and

$\pi(S^0, \{S^1,...,S^k\}) = (\pi(S^0), \{\pi(S_1),...,\pi(S^k)\})$,