The effective potential (Taylor's equation 8.32) is

$$U_{\text{eff}} = \frac{-Gm_e m_s}{r} + \frac{\ell^2}{2\mu r^2}$$

There is an equilibrium at the minimum of U_{eff}. So we should solve $\frac{d}{dr}U_{\text{eff}} = 0$ for r, and call this r_0, the equilibrium distance.

In [8]:
```
var('G m_e m_s ell mu')
Ue(r)=-G*m_e*m_s/r+ell^2/(2*mu*r^2)
show(Ue(r))
```

Out[8]: $$-\frac{G m_{e} m_{s}}{r} + \frac{\mathit{ell}^{2}}{2 \, \mu r^{2}}$$

In [6]:
```
show( diff(Ue(r),r))
show(solve(    diff(Ue(r),r)==0, r))
r_0=ell^2/(G*m_e*m_s*mu)
```

Out[6]: $$\frac{G m_{e} m_{s}}{r^{2}} - \frac{\mathit{ell}^{2}}{\mu r^{3}}$$
$$\left[r = \frac{\mathit{ell}^{2}}{G m_{e} m_{s} \mu}\right]$$

There is minimum of U_{eff} at $r = r_0 = \frac{\ell^2}{Gm_e m_s \mu}$.

The "spring constant" is $k = \frac{d^2}{dr^2}U_{\text{eff}}\Big|^{r=r_0}$, and the angular frequency of small oscillations is $\omega \approx \sqrt{k/m_e}$.

In [7]:
```
# I'll name the second derivative of U as Udp(r), that is, "U double prime"
Udp(r)=diff(Ue(r),r,2)
show('Udp(r)=',Udp(r))
show('Udp(r_0)=k=',Udp(r_0))
```

Out[7]: $$\verb|Udp(r)=| -\frac{2 \, G m_{e} m_{s}}{r^{3}} + \frac{3 \, \mathit{ell}^{2}}{\mu r^{4}}$$
$$\verb|Udp(r_0)=k=| \frac{G^{4} m_{e}^{4} m_{s}^{4} \mu^{3}}{\mathit{ell}^{6}}$$

OK, let's now start to approximate $\mu \approx m_e$ so this result is

$$k \approx \frac{G^4 m_e^7 m_s^4}{\ell^6}$$

and therefore, the angular frequency of oscillations is

$$\omega \approx \sqrt{\frac{k}{m_e}} = \sqrt{\frac{G^4 m_e^6 m_s^4}{\ell^6}} = \frac{G^2 m_e^3 m_s^2}{\ell^3}.$$

Now, cast an eye at equation 8.23 which is just re-arranging the formula for angular momentum $\ell = mr^2\dot{\phi}$:

$$\dot{\phi} = \frac{\ell}{\mu r^2}.$$

For the earth-sun system, when $r = r_0$, the angular speed is constant. Let's write this as $\dot{\phi} = \omega_e$. And, approximating $\mu \approx m_e$:

$$\omega_e = \frac{\ell}{m_e r_0^2} \qquad (*)$$

We would love to show that $\omega = \omega_e$. But how? The key is to re-examine our solution for the equilibrium separation, r_0 which connects r_0 to the angular momentum ℓ, and the gravitational constant G. It was

$$r_0 = \frac{\ell^2}{Gm_e m_s \mu} \approx \frac{\ell^2}{Gm_e^2 m_s}$$

$$\Rightarrow G^2 m_e^4 m_s^2 = \frac{\ell^4}{r_0^2}$$

Now, re-examining the formula for ω, (and using equation (*)):

$$\omega = \frac{G^2 m_e^3 m_s^2}{\ell^3} \tag{1}$$

$$= G^2 m_e^4 m_s^2 \cdot \frac{1}{m_e \ell^3} \tag{2}$$

$$= \frac{\ell^4}{r_0^2} \cdot \frac{1}{m_e \ell^3} = \frac{\ell}{m_e r_0^2} = \omega_e \tag{3}$$

In [0]:

```

```