2.4 The Precise Definition of a Limit
Math 1271, TA: Amy DeCelles

1. Overview

Definition of a Limit

We say “the limit of \( f(x) \) as \( x \) approaches \( a \) is \( L \)” if the following condition is satisfied:

\[
\text{For every number } \epsilon > 0 \text{ there is a number } \delta > 0 \text{ such that:}
\]
\[
\text{if } |x - a| < \delta \text{ then } |f(x) - L| < \epsilon
\]

Parsing this definition

Our intuitive understanding is that \( L \) is the limit (of \( f(x) \) as \( x \to a \)) if \( f(x) \) gets closer and closer to \( L \) as \( x \) gets closer and closer to \( a \). Let’s see how this matches up with the precise definition.

First look at the expression \( |x - a| \). This is the distance between \( x \) and \( a \). So if we make \( \delta \) smaller and smaller, that means that \( x \) is getting closer and closer to \( a \). Similarly, \( |f(x) - L| \) is the distance between the \( y \)-values \( f(x) \) and \( L \), so if \( \epsilon \) gets smaller and smaller, that means that \( f(x) \) is getting closer and closer to \( L \). So, just to make this clear: \( \delta \) is a distance that specifies an \( x \)-range: how far away from \( a \) can \( x \) be? ... it must be within \( \delta \) units of \( a \). We’ll call this \( x \)-range a “\( \delta \)-neighborhood” of \( a \). And \( \epsilon \) is a distance that specifies a \( y \)-range: how far away from \( L \) can \( f(x) \) be? ... it will be within \( \epsilon \) units of \( L \). This \( y \)-range we’ll call an \( \epsilon \)-neighborhood of \( L \).

So when we say that:

\[
\text{if } |x - a| < \delta \text{ then } |f(x) - L| < \epsilon
\]

that is like saying:

If \( x \) is close enough to \( a \) (namely in a \( \delta \)-neighborhood of \( a \)) then \( f(x) \) is guaranteed to be close to \( L \) (namely in an \( \epsilon \)-neighborhood of \( L \)).

Ok, well then, what is the deal with the “for every \( \epsilon > 0 \) there is a \( \delta > 0 \)” part? This means that no matter how small you make the \( \epsilon \)-neighborhood of \( L \), you will always be able to find a \( \delta \)-neighborhood of \( a \), that “works,” i.e. a \( \delta \)-neighborhood small enough to guarantee that the \( y \)-values of the graph of \( f \) are in the \( \epsilon \)-neighborhood of \( L \). The point is that we can get \( f(x) \) infinitely close to \( L \), by just making the \( \delta \)-neighborhood of \( a \) smaller and smaller.

2. Example

Problem: Prove, using the \( \epsilon \)-\( \delta \) definition of limit that:

\[
\lim_{x \to 1} 5x - 3 = 2
\]

Solution:

We need to show:

For any \( \epsilon > 0 \), there is a \( \delta > 0 \) such that:

\[
\text{if } |x - 1| < \delta \text{ then } |(5x - 3) - 2| < \epsilon
\]
Before we write our proof, we need to do some thinking. (This is like the prewriting you would do before writing a paper.) We treat $\epsilon$ like a fixed number. We want to figure out what $\delta$ will work, given the $\epsilon$ we have. We start with the $\epsilon$ condition:

$$|(5x - 3) - 2| < \epsilon$$

Simplifying, we get:

$$|5x - 5| < \epsilon$$

We factor out a 5:

$$5 \cdot |x - 1| < \epsilon$$

So if we rewrite what we have to show, using the simplification we just did, we get:

For any $\epsilon > 0$, there is a $\delta > 0$ such that:

$$\text{if } |x - 1| < \delta \text{ then } 5 \cdot |x - 1| < \epsilon$$

Well, if $|x - 1| < \delta$ then $5 \cdot |x - 1| < 5\delta$. So we just need to have $5\delta \leq \epsilon$. So we will choose $\delta = \frac{\epsilon}{5}$.

Now that we have figured out what $\delta$ will work, we need to go back and write up an argument. (This is like writing a paper: we take the work we just did and arrange it nicely to construct an argument.)

**Proof:**

Given any $\epsilon > 0$, we can define $\delta = \frac{\epsilon}{5}$. Then:

$$\text{if } |x - 1| < \delta \text{ then } |x - 1| < \frac{\epsilon}{5}$$

then $5 \cdot |x - 1| < 5 \cdot \frac{\epsilon}{5}$

then $5 \cdot |x - 1| < \epsilon$

But $5 \cdot |x - 1| = |(5x - 3) - 2|$, so we have shown:

$$\text{if } |x - 1| < \delta \text{ then } |(5x - 3) - 2| < \epsilon$$

So we have shown:

For any $\epsilon > 0$, there is a $\delta > 0$ such that:

$$\text{if } |x - 1| < \delta \text{ then } |(5x - 3) - 2| < \epsilon$$

and we are done!