3.2 The Product Rule and the Quotient Rule
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1. Overview

You need to memorize the product rule and the quotient rule. And, more than that actually: you need to internalize them. The best way to do that is just by practicing until you can use them without even thinking about it.

**Product Rule:**

\[(fg)' = f'g + fg'\]

In words: The derivative of a product of two functions is: the derivative of the first, times the second, plus the first times the derivative of the second.

**Quotient Rule:**

\[\left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2}\]

In words: The derivative of a quotient of two functions is: the derivative of the top, times the bottom, minus the top times the derivative of the bottom, all over the bottom squared.

2. Examples

1.) Find the derivative of

\[g(x) = \sqrt{x} e^x\]

Since \(g(x)\) is the product of two functions, we use the product rule:

\[g'(x) = (\sqrt{x})'(e^x) + (\sqrt{x})(e^x)'\]

Remembering that \(\sqrt{x} = x^{1/2}\), we compute the derivatives:

\[g'(x) = \left(\frac{1}{2}x^{-1/2}\right)(e^x) + (x^{1/2})(e^x)\]

And, simplifying, we get:

\[g'(x) = \left(\frac{1}{2\sqrt{x}} + \sqrt{x}\right)e^x\]

2.) Find the derivative of

\[y = \frac{\sqrt{x} - 1}{\sqrt{x} + 1}\]

Since \(y\) is the quotient of two functions, we use the quotient rule:

\[y' = \frac{(\sqrt{x} - 1)'(\sqrt{x} + 1) - (\sqrt{x} - 1)(\sqrt{x} + 1)'}{(\sqrt{x} + 1)^2}\]

Converting all the square roots to powers:

\[y' = \frac{(x^{1/2} - 1)'(x^{1/2} + 1) - (x^{1/2} - 1)(x^{1/2} + 1)'}{(x^{1/2} + 1)^2}\]

Now we take derivatives:

\[y' = \frac{(\frac{1}{2}x^{-1/2})(x^{1/2} + 1) - (x^{1/2} - 1)(\frac{1}{2}x^{-1/2})}{(x^{1/2} + 1)^2}\]
And we simplify the numerator:

\[
y' = \frac{\left(\frac{1}{2} + \frac{1}{2}x^{-1/2}\right) - \left(\frac{1}{2} - \frac{1}{2}x^{-1/2}\right)}{(x^{1/2} + 1)^2} \\
= \frac{\frac{1}{2} + \frac{1}{2}x^{-1/2} - \frac{1}{2} + \frac{1}{2}x^{-1/2}}{(x^{1/2} + 1)^2} \\
= \frac{x^{-1/2}}{(x^{1/2} + 1)^2}
\]

Rewriting in terms of the square roots:

\[
y' = \frac{1}{\sqrt{x} (\sqrt{x} + 1)^2}
\]