4.1 Maximum and Minimum Values

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1. Overview

Outline:

1. Definition of absolute and local maximum and minimum values of a function
2. Theorem: Local max/min values will always occur at a critical number, i.e. a number c in the domain of f such that \( f'(c) = 0 \) or \( f'(c) \) does not exist.
3. Theorem: A continuous function will always have an absolute max value and an absolute min value on a closed interval.
4. Finding abs max/min values (for a continuous function on a closed interval)
   - Either the max/min value occurs at an endpoint, or it occurs somewhere in the middle.
   - If it occurs in the middle, it has to be a local max/min value as well, and we know that local max/min values always occur at critical numbers.
   - So the abs max/min will be either at an endpoint or a critical number.
   - You just have to check the value of the function at the endpoints and the critical numbers and see what value is the greatest and which the least; those are your absolute max and min values.

Note: We will learn how to find local max/min values in section 4.3.

You should be able to look at a graph of a function and determine:

1. the critical numbers
2. the local min/max values
3. the absolute min/max values

2. Examples

1.) Find the critical numbers of

\[ g(x) = \sqrt{1 - x^2} \]

Remember that the critical numbers of \( g \) are numbers in the domain of \( g \) where \( g'(x) = 0 \) or \( g'(x) \) DNE. So we start by taking the derivative (have to use the chain rule):

\[ g'(x) = \frac{1}{2} (1 - x^2)^{-1/2} \cdot (-2x) = -\frac{x}{\sqrt{1 - x^2}} \]

When is \( g'(x) = 0 \)? Just when \( x = 0 \) (because the numerator will be zero, and the denominator will be nonzero.)

When does \( g'(x) \) DNE? Well, \( g'(x) \) DNE if the denominator is zero, i.e. if \( x = \pm 1 \), and \( g'(x) \) also DNE if the inside of the square root is negative, i.e. if:

\[
\begin{align*}
1 - x^2 &< 0 \\
1 &< x^2 \\
\sqrt{1} &< |x| \\
1 &< |x| \\
x &> 1 \quad \text{or} \quad x < -1
\end{align*}
\]
So \( g'(x) \) DNE if \( x \geq 1 \) or \( x \leq -1 \).

To sum up: \( g'(x) = 0 \) for \( x = 0 \) and \( g'(x) \) DNE for \( x \geq 1 \) or \( x \leq -1 \). Now we just need to check which of these numbers are in the domain of \( g \). The domain of \( g \) is going to be all real numbers, except those numbers that make the inside of the square root negative. We’ve already figured out that the square root is negative when \( x > 1 \) or \( x < -1 \), so the domain of \( g \) is \([-1, 1]\). So the critical numbers are \(-1, 0, \) and \(1).

2.) Find the absolute max/min values of \( f(x) = x^3 - 3x + 1 \) on the interval \([0, 3]\).

We know that the absolute max/min values of \( f(x) \) will occur either at an endpoint or a critical number. So we start by finding the critical numbers. We need to take the derivative:

\[
 f'(x) = 3x^2 - 3 = 3(x^2 - 1)
\]

Where is \( f'(x) = 0 \)? If we factor \( f'(x) \),

\[
 f'(x) = 3(x - 1)(x + 1)
\]

we can see that \( f'(x) = 0 \) when \( x = \pm 1 \). Where does \( f'(x) \) DNE? Nowhere. Since the numbers \(\pm 1\) are in the domain of \( f \), \(\pm 1\) are the critical numbers of \( f(x) \).

So we need to check the value of \( f(x) \) at each endpoint of \([0, 3]\) and each critical number in \([0, 3]\). Since \(-1\) is not in the interval, we don’t have to check \( f(x) \) there.

\[
 f(0) = 0 - 0 + 1 = 1 \\
 f(1) = 1 - 3(1) + 1 = -1 \\
 f(3) = 3^3 - 3(3) + 1 = 27 - 9 + 1 = 19
\]

So the absolute max value is 19 and the absolute min value is -1.

3.) Find the absolute max/min values of \( f(x) = \frac{x^2 - 4}{x^2 + 4} \) on the interval \([-4, 4]\).

We know that the absolute max/min values of \( f(x) \) will occur either at an endpoint or a critical number. So we start by finding the critical numbers. We take the derivative using the quotient rule:

\[
 f'(x) = \frac{(2x)(x^2 + 4) - (x^2 - 4)(2x)}{(x^2 + 4)^2} \\
 = \frac{2x(x^2 + 4) - (x^2 - 4)}{(x^2 + 4)^2} \\
 = \frac{2x(x^2 + 4 - x^2 + 4)}{(x^2 + 4)^2} \\
 = \frac{16x}{(x^2 + 4)^2}
\]

We can see that \( f'(x) = 0 \) just when \( x = 0 \), and that there are no values of \( x \) where \( f'(x) \) DNE. Since \( x = 0 \) is in the domain of \( f \), it is a critical number.
So we need to check the value of \( f(x) \) at each endpoint of \([-4, 4]\) and each critical number in \([-4, 4]\):

\[
\begin{align*}
  f(-4) &= \frac{0}{8} = 0 \\
  f(0) &= \frac{-4}{4} = -1 \\
  f(4) &= \frac{0}{8} = 0
\end{align*}
\]

So the absolute max value is 0 and the absolute min value is \(-1\).

**Note:** There are two places where the absolute max value is attained, \( x = -4 \) and \( x = 4 \). This is not a problem. The value \( y = 0 \) is still the highest value that \( f(x) \) achieves on the interval.