Consider the matrices $A$ and $B$ below. The matrices $R_A$ and $R_B$ are the reduced row echelon forms of $A$ and $B$ respectively.

$$A = \begin{pmatrix} 1 & 2 & 1 & 3 & 1 \\ -1 & 1 & 2 & 5 & 4 \\ 2 & -1 & -3 & 1 & 2 \\ 3 & 4 & 1 & 1 & -3 \end{pmatrix} \quad R_A = \begin{pmatrix} 1 & 0 & -1 & 0 & 0 \\ 0 & 1 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$B = \begin{pmatrix} 1 & 2 & 1 & 3 & 1 \\ -1 & 1 & 2 & 5 & 4 \\ 2 & -1 & -3 & 1 & 2 \\ 3 & 4 & 1 & 1 & 2 \end{pmatrix} \quad R_B = \begin{pmatrix} 1 & 0 & -1 & 0 & 0 \\ 0 & 1 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

The purpose of the exercise is to determine what information $R_A$ (and $R_B$) tell us about the subspaces associated with $A$ (resp. $B$).

1. Recall that the elementary row operations do not change the span of the row vectors of a matrix.

   (a) What does this imply about the row space of $A$ and the row space of $R_A$ (or the row space of $B$ and the row space of $R_B$)?

   (b) Use this information to find a basis for the row space of $A$ and the row space of $B$.

2. Recall that elementary row operations do not change the dependency relations among the columns of a matrix. (That is why we can use the reduced row echelon form to determine dependency relations among column vectors.)

   (a) Look at the column vectors of $R_A$ and choose a basis for the column space of $R_A$ from among those columns.

   (b) Use this information to determine a basis for the column space of $A$.

   (c) Use the same strategy to determine a basis for the column space of $B$.

3. Recall that the null space of a matrix $M$ is the set of all vectors $v$ such that $Mv = 0$. Applying elementary row operations to the augmented matrix $(M\mid 0)$ is equivalent to multiplying both sides by elementary matrices.

   (a) What does this imply about the null space of a matrix and the null space of its reduced row echelon form?

   (b) Describe the vectors in the null space of $R_A$ entry-wise. Determine a basis for the null space of $R_A$ and the null space of $A$.

   (c) Use the same strategy to determine a basis for the null space of $B$.

4. What are the dimensions of the row spaces, column spaces, and null spaces of $A$ and $B$? What pattern(s) do you notice? Will these patterns hold in general? Why or why not?