Let \( f(x) \) be the step function on \([-\pi, \pi]\\):

\[
f(x) = \begin{cases} 
1 & \text{if } x > 0 \\
-1 & \text{if } x < 0 
\end{cases}
\]
We have shown that the set $V$ of bounded functions on the closed interval $[-\pi, \pi]$ that are piecewise continuous is a vectorspace, with an inner product given by

$$\langle f, g \rangle = \int_{-\pi}^{\pi} f(x) g(x) \, dx$$

Further, with respect to this inner product, the functions

$$1, \sin x, \cos x, \sin 2x, \cos 2x, \ldots, \sin kx, \cos kx, \ldots$$

are mutually orthogonal in $V$, and

$$\langle \sin kx, \sin kx \rangle = \pi \quad \langle \cos kx, \cos kx \rangle = \pi \quad \langle 1, 1 \rangle = 2\pi$$

This orthogonal set of vectors (functions) spans $V$ in the sense that any vector (function) in $V$ can be approximated arbitrarily well by linear combinations of vectors in this orthogonal set.
Let $W_0$ be the subspace of $V$ spanned by $1$, $W_1$ be the subspace spanned by $1$, $\sin x$, and $\cos x$, and so on:

$$W_0 = \text{span}(\{1\})$$
$$W_1 = \text{span}(\{1, \sin x, \cos x\})$$
$$W_k = \text{span}(\{1, \sin x, \cos x, \sin 2x, \cos 2x, \ldots, \sin kx, \cos kx\})$$
The zeroth approximation is:

\[ f_0 = \text{Proj}_{W_0}f = \text{Proj}_1f = \frac{\langle f, 1 \rangle}{\langle 1, 1 \rangle} 1 = \frac{0}{2\pi} 1 = 0 \]

The first approximation is:

\[ f_1 = \text{Proj}_{W_1}f \]

\[ = \text{Proj}_1f + \text{Proj}_{\cos x}f + \text{Proj}_{\sin x}f \]

\[ = \frac{\langle f, 1 \rangle}{\langle 1, 1 \rangle} \cdot 1 + \frac{\langle f, \cos x \rangle}{\langle \cos x, \cos x \rangle} \cdot \cos x + \frac{\langle f, \sin x \rangle}{\langle \sin x, \sin x \rangle} \cdot \sin x \]

\[ = 0 + 0 + \frac{4}{\pi} \sin x \]
The first approximation is

\[ f_1(x) = \frac{4}{\pi} \sin x \]
In general, the $N$th approximation is:

$$f_N = \text{Proj}_{W_N} f = \text{Proj}_1 f + \sum_{k=1}^{N} \left( \text{Proj}_{\sin kx} f + \text{Proj}_{\cos kx} f \right)$$

For the step function,

$$\text{Proj}_1 f = 0$$

$$\text{Proj}_{\cos kx} = 0 \quad \text{(for all } k > 0)$$

$$\text{Proj}_{\sin kx} = 0 \quad \text{(for even } k > 0)$$

$$\text{Proj}_{\sin kx} = \frac{4}{\pi k} \sin kx \quad \text{(for odd } k > 0)$$
So the second approximation is the same as the first, but the third approximation is

\[ f_3 = \frac{4}{\pi} \sin x + \frac{4}{3\pi} \sin(3x) \]
Writing odd $k$ as $k = 2n + 1$, we can see that the Fourier series for the step function is:

$$\sum_{n=0}^{\infty} \frac{4}{\pi(2n + 1)} \sin ((2n + 1)x)$$

This [movie](#) shows the first twenty approximations.