

3.9 Related Rates

Math 1271, TA: Amy DeCelles

1. Overview

In related rates problems you start with a relationship between two quantities (for example the volume and the radius of a sphere) and you use implicit differentiation to come up with a relationship between the rates (the rate at which the volume is changing and the rate at which the radius is changing.)

$$\text{Quantities} \xrightarrow{\frac{d}{dx}} \text{Rates}$$

Here's a procedure for solving these problems:

1. **Set-up** Write down what you know, draw a picture, name some variables, write down what it is that you want to find.
2. **Quantities and Rates** Draw a picture of the action, a snapshot at arbitrary time t . What are the quantities (i.e. the things that are changing)? What are the rates (i.e. $\frac{d}{dt}$ of the quantities)?
3. **Equation with Quantities** Find an equation relating the quantities (at arbitrary time t , not any specific moment). Use anything in your mathematical toolkit, like equations for volume and surface area, the pythagorean theorem, similar triangles, special triangles, etc.
4. **Implicit Differentiation** Take $\frac{d}{dt}$ of the equation with quantities to get an equation with rates. Remember to use implicit differentiation!
5. **Plug in** Now you can look at whatever specific moment you're interested in. Plug in and solve for the rate you want.
6. **Check** Check to see that you've answered the question. (Did I include units? Do I have the right sign?)

2. Examples

1.) A spherical balloon is being deflated at a rate of $60\text{cm}^3/\text{min}$. How fast is the diameter decreasing when the diameter is 10cm ?

What are the quantities and rates? One rate is given to us in the problem: $60\text{cm}^3/\text{min}$. We have to figure out what *quantity* is changing at that rate. (The units can be a clue!) It is the volume of air in the balloon. So if we say that V is the volume of air in the balloon at any given time, then $\frac{dV}{dt} = -60$. (It is negative, because it is decreasing.)

What rate do we want to find? We want to find the rate at which the diameter is decreasing. So if we let x be the diameter of the balloon at any given time, then we need to find $\frac{dx}{dt}$. So the quantities are V and x and the rates are $\frac{dV}{dt}$ and $\frac{dx}{dt}$. We want to find an equation relating the quantities, i.e. an equation with V and x . We remember the volume of a sphere is:

$$V = \frac{4}{3}\pi r^3$$

and the radius is half the diameter, so

$$V = \frac{4}{3}\pi\left(\frac{1}{2}x\right)^3 = \frac{4}{3}\pi\left(\frac{1}{8}\right)x^3 = \frac{\pi}{6}x^3$$

(At this point you might be tempted to plug in $x = 10$. But you shouldn't do that because that freezes the action. If you plug in $x = 10$ right now then you're acting as if x doesn't change, and if x doesn't change then the rate at which it is decreasing is just zero. So wait until after we've differentiated before plugging in.)

To get an equation relating the rates, we differentiate (with respect to t) the equation relating the quantities.

$$\begin{aligned}V &= \frac{\pi}{6}x^3 \\ \frac{d}{dt}(V) &= \frac{d}{dt}\left(\frac{\pi}{6}x^3\right) \\ \frac{dV}{dt} &= \frac{\pi}{6} \cdot 3x^2 \cdot \frac{dx}{dt}\end{aligned}$$

Notice that we had to use the chain rule on the right hand side, because x is a function of t . Now we have a function relating the rates.

Now we can look at the specific moment we're interested in, when the diameter is 10cm. We plug in what we know, i.e. we know that $\frac{dV}{dt} = -60$ and $x = 10$:

$$-60 = \frac{\pi}{6} \cdot 3(10)^2 \cdot \frac{dx}{dt} = \frac{\pi}{6} \cdot 300 \cdot \frac{dx}{dt} = 50\pi \frac{dx}{dt}$$

So $\frac{dx}{dt}$ is:

$$\frac{dx}{dt} = -\frac{60}{50\pi} = -\frac{6}{5\pi}$$

The last thing we should do is make sure we answer the question. The question asked us how fast the diameter was decreasing, whereas we have just computed the rate at which it was changing. So we need to take off the negative sign. Also we need to mention the correct units. Since the rate for the volume was given in cm^3/min , and the units for the rate of change of the diameter will be cm/min . So we conclude:

When the diameter of the balloon is 10 cm, the diameter is decreasing at a rate of $\frac{6}{5\pi}$ cm/min.