Oligopoly In Partition Function Form*

Daniel S. Arthur

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Abstract:

This paper offers a framework for examining coalitional formation in Oligopoly using the game theoretic partition function form. Firms producing a homogeneous product and having known capacity limits are examined. In determining whether two coalitions should join together in various scenarios, the results suggest that if the initial number of coalitions producing the optimal quantity not at capacity is less than a determinable number, then the two coalitions should join. A four player example is included.

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I. Introduction and Motivation

Oligopoly models have been studied since Augustin Cournot first introduced his mathematical model in 1838. In more recent years, oligopoly has been modeled in the game-theoretic framework. The two approaches to modeling oligopoly in the game-theoretic framework are the noncooperative and the cooperative. Under the noncooperative approach, the oligopolists act independently and attempt to maximize their own profits while taking into account the actions of their fellow oligopolists. The cooperative approach is based on the assumption that the oligopolists can make binding agreements and collude. Thus, through cooperation, the oligopolists can determine the coordinated policy that will yield the highest profits. The issue to be addressed by the colluding oligopolists and the goal of this paper is the determination of the fair division of profits that should be agreed upon.

In the field of study on oligopoly, the cooperative approach is often regarded as unrealistic. The reason being that there exist antitrust laws preventing legally binding agreements between firms. Therefore, firms cannot enforce their agreements and it is easily shown that it is not in the best interest of any firm in a collusion to keep its word. Thus the agreement cannot be considered believable. In the static, or one period model, this argument is certainly valid and points to the noncooperative Cournot equilibrium as the only realistic outcome. However, the situation changes when time is introduced and the dynamic, or many period, model is analyzed. As Fudenberg and Tirole [1986] comment upon, the historical actions of the firms in the market are observed by each other over time and realistic motivations for adhering to agreements exist. Horizontal mergers also offer a legal single period scenario for study. Therefore, the unrealistic single-period assumption is made and analyzed in this paper.

The partition function form was first introduced in 1963 by Lucas and Thrall. The main consideration of this function form is the partition, or set of coalitions, that the players of the game form. The worth of a coalition is dependent on how the rest of the firms collude. Therefore a coalition can have several worths, namely a worth for each permutation of the players not in the coalition. This aspect of the partition function form seems to encompass the environment in which oligopolies operate and is the motivation for creating a mathematical model of oligopoly in partition function form.

II. Economic and Game Theoretic Definitions

(1) Consumer's Utility Function: $u = aq - \frac{bq^2}{n} - pq$ where u is the monetary measure of total benefit to consumers from q units of the commodity, a and b are positive constants, n is the number of firms in the industry, and p is the price of the commodity.

(2) Demand Function: $p = a - \frac{2bq}{n}$ where p is price that will induce the consumers to purchase q units of the commodity thus maximizing the consumer's utility, a and b are the same positive constants, and n is once again the number of firms in the industry.

(3) Cost Function: $C_i(q_i) = c_i q_i$ where c_i is the cost of producing one unit by firm i, and q_i is the quantity produced by firm i. In the effort of simplification, the assumption of equal average costs is made. Therefore, $c_i = c \ \forall i \in \{1, \ldots, n\}$ and $C_i(q_i) = cq_i$. The assumption of a linear cost function guarantees a q_i^* such that profits for firm i are maximized.

(4) Profit Function: $\Pi_i = bq_i\left(A - \frac{2q}{n}\right)$ where $A = \frac{a-c}{b}$ where a,b are the constants from the demand function, c is the cost to produce one unit, and q_i is the quantity produced by firm i.

(5) Capacity Restriction: $K(N) = \sum_{i=1}^{n} k_i$ where $K(N)$ is the capacity of the industry, and k_i is the capacity of firm i.

Without loss of generality, we can assume:

$$k_1 \geq \cdots \geq k_n \quad \text{and} \quad \sum_{i=1}^{n} k_i \leq \frac{nA}{2}$$

The quantity $\frac{nA}{2}$ is the quantity that will equate the market price with the cost to produce the item, thus profits are equal to zero. The first restriction is designed to introduce some type of ordering to the firms in the industry. The second restriction makes sense because the industry as a whole should not spend extra money to allow for an amount of capacity that will drive the price below cost. It is worth noting that if $\sum_{i=1}^{n} k_i \leq \frac{nA}{4}$, which is the optimal capacity for a monopoly, then each firm would be producing at capacity in every possible coalition.

The following game theoretic definitions are used to model oligopoly in partition function form.

$N=\{1,2,\ldots,n\}$ is the set of players in a n person game.

$CL=\{S_j \mid S_j \subseteq N,\ S_j \neq \emptyset\}$ is the set of coalitions of N.

PT is the set of partitions of N: $\{S_1,\ldots,S_r\} \in PT$ if and only if $S_1 \cup \ldots \cup S_r=N$, for all j such that $S_j \neq \emptyset$, for all k such that $S_k \cap S_j=\emptyset$ if $j \neq k$.

$v(S_j;P)$ is the worth of coalition S_j given partition $P=\{S_1, S_2, \ldots, S_r\}$.

The above definition places this oligopoly model into the framework of a cooperative game. The worth of coalition S_j should be impacted greatly by the coalitions formed by the other $N-S_j$ players. Thus the above definition states that $v(S_j;P)$ is the joint profit of coalition S_j in the partition game when each coalition in P acts as a single player. In other words, the worth of coalition S_j is found by maximizing $\pi(S_j;P)=\sum_{i \in S_j} \pi_i$ given that the coalitions formed by the other $N-S_j$ players are also maximizing their respective joint profits.

For any given partition, there are a total of r coalitions such that $1 \leq r \leq n$. Because of the ordering of capacities assumption, coalitions $1,\ldots,m$ are producing the optimal quantity not at capacity, while coalitions $m+1,\ldots,r$ are producing at capacity, such that $0 \leq m \leq r$.

In order to obtain the q_i^* which will yield the Π_i^* which is $v(i;Q)$, the maximum profit or value for player i, partial derivatives will be taken and set equal to zero.

In the effort to simplify the rigorous calculations involved, an additional definition will be introduced: $h_i=\frac{k_i}{A}$. Thus, since A is a constant, h_i is really nothing more than an adjusted capacity of firm i. Also, let $h(S_j)=\sum_{i \in S_j} h_i$, where $h(S_j)$ is the adjusted capacity of coalition S_j.

III. The n-player Game

Each firm in the industry has a capacity k_i. Therefore, each coalition has the capacity $k(S_j)=\sum_{i \in S_j} k_i$. The first derivative of (4) set equal to zero yields:

(6) $q(S_i)+q(N)=\frac{nA}{2}$,

where $q(\mathrm{N})$ is the total quantity produced by the industry.

Thus the optimal quantity for coalition S_i to produce is:

$$(7)\quad q^*(S_i;P)=\min\left\{\frac{nA}{4}-\frac{1}{2}\sum_{j\neq i}q(S_j)\,,\;k(S_i)\right\}.$$

Summing these coalitions yields:

$$(8)\quad \sum_{i=1}^{m}\left(q(S_i)+q(N)=\frac{nA}{2}\right)$$

$$(9)\quad \sum_{i=m+1}^{r}\left(q(S_i)=k(S_i)\right)$$

Adding these two summations together yields:

$$(10)\quad q^*(N)=\frac{mnA}{2(m+1)}+\frac{1}{(m+1)}\left(\sum_{i=m+1}^{r}k(S_i)\right)$$

Substituting (10) into (6) yields:

$$(11)\quad q^*(S_i;P)=A\left(\frac{n}{2}-\frac{mn}{2(m+1)}-\frac{1}{(m+1)}\left(\sum_{j=m+1}^{r}h(S_j)\right)\right)$$ where $q^*(S_i;P)$ is the optimal quantity for coalition S_j given the partition $P=\{S_1,S_2,\ldots,S_r\}$.

Substituting (10) and (11) into (4) yields:

$$(12)\quad \pi^*(S_i;P)=\frac{2bA^2}{n(m+1)^2}\left(\frac{n}{2}-\sum_{j=m+1}^{r}h(S_j)\right)^2,\; q^*(S_i;P)=A\left(\frac{n}{2}-\frac{mn}{2(m+1)}-\frac{1}{(m+1)}\left(\sum_{i=m+1}^{r}h_i\right)\right),$$

if $i=1,\ldots,m$;

$$(13)\quad \pi^*(S_i;P)=\frac{2bA^2}{n(m+1)}h(S_i)\left(\frac{n}{2}-\sum_{j=m+1}^{r}h(S_j)\right),\; q^*(S_i;P)=k(S_i),\text{ if } i=m+1,\ldots,r;$$

where m is the largest integer for which $q^*(S_i;P)\leq k(S_i)$ for all $i=1,\ldots,r$.

The assumption of capacity restrictions plays a key role in the determination of whether a group of firms collude. Of course, the members of the coalition must have a combined capacity level that will allow for the true optimal level of production to be reached, but the capacities of all the other firms are of great importance.

IV. Coalitional Formation Results Using Superadditivity

When a group of firms merge or collude, they act as one entity, or player. Thus the levels of production that are optimal for each coalition are a form of Cournot equilibrium on a reduced player game where r is the number of players, which are coalitions, and n is the number of firms, or potential players, in the industry. It is easily shown that when the number of coalitions in the partition is reduced, the aggregate quantity for the industry is also reduced. Because the optimal quantity for a coalition is dependent on the number of coalitions in the partition and not the actual number of firms in any given coalition, coalitions of different sizes are producing the same optimal quantity. Thus each coalition is getting the same profit which is to be divided between the members of the coalition. Therefore, if for example, there are two coalitions in the partition, the first with just one player, and the other with $n-1$ players. Each coalition, without capacity restrictions, will have the same optimal quantity and thus the same profit. However, the one player in the one-player coalition receives all the profit while the $(n-1)$-player coalition must divide the profit between its $n-1$ players. If the $n-1$ players are not receiving at least as much as they could get by not being a member of the coalition, they will not join. However, because capacity restrictions are being entered into the game, when the capacity level of the smaller coalitions is below a certain amount, it becomes profitable for coalitions to form out of the remaining players, because the smaller coalitions cannot expand their output to the desired level. Next the issue of superadditivity will be discussed.

A game is v-superadditive if given coalitions S_i, S_j and partition P:

$$v(S_i;P)+v(S_j;P)\leq v(S_i\cup S_j;\{P-\{S_i,S_j\}\}\cup\{S_i\cup S_j\}) \text{ for all } (S_i;P),(S_j;P)\in ECL$$

where ECL is the set of embedded coalitions: $\{(S;P)\mid S\in P\in PT\}$

The above definition examines the case of two coalitions joining together. This format is applicable because when coalitions form, this formation process can be thought of as first two individual firms joining together, then one more firm or another coalition of two firms joining them, until the grand coalition is formed. In the case of leveraged buyouts, the buying group purchases one firm at a time. In the case of joint ventures, granted there are times when three or more coalitions join, but this process can also be reasonably thought of as two joining first, then a third party joining the first coalition and so on.

Then the question to be answered is when would S_i and S_j join together. There are three cases to be examined. The first case is when S_i and S_j are both presently producing the optimal quantity not at capacity. The second case is when one of the two coalitions is producing the optimal quantity and the other is presently producing at capacity. The third case is when both S_i and S_j are presently producing at capacity.

(i) Case 1, S_i and S_j not at capacity

From (12) and (13) derived earlier, the inequality that needs to be solved is,

$$(14) \quad \frac{2}{(m+1)^2}\left(\frac{n}{2}-2H+\frac{2}{n}H^2\right) \leq \frac{1}{((m+1)-1)^2}\left(\frac{n}{2}-2H+\frac{2}{n}H^2\right)$$

where $H = \sum_{t=m+1}^{r} h(S_t)$

Note that H is a constant because H simply represents the combined quantity of all coalitions that are producing at capacity and that will still be after S_i and S_j join together. Thus the inequality that needs to be solved is:

$$(15) \quad \frac{2}{(m+1)^2} \leq \frac{1}{m^2}$$

which yields:

$$(16) \quad m \leq \frac{1}{\sqrt{2}-1} \approx 2.4142$$

Since m, the number of coalitions producing the optimal quantity not at capacity, can only be an integer value. The above result is a sufficiency condition that can be interpreted as the following. If S_i and S_j are the only coalitions producing the optimal quantity not at capacity, then by joining together, they will be more

profitable.

(ii) Case 2, S_i not at capacity, S_j at capacity:

Once again, from (12) and (13) derived earlier, the inequality that must be solved is:

$$(17) \quad \frac{1}{(m+1)^2}\left(\frac{n}{2}-2H+\frac{2}{n}H^2\right)+\frac{1}{m+1}\left(h(S_j)-\frac{2h(S_j)H}{n}\right) \leq \frac{1}{(m+1)^2}\left(\frac{n}{2}-2(H-h(S_j))+\frac{2}{n}(H-h(S_j))^2\right)$$

Where $H = \sum_{t=m+1}^{r} h(S_t)$ from the original partition, before S_i and S_j join.

The algebra yields (see appendix I.):

(18) $$m \le \frac{n - 2H + h(S_j)}{n - 2H}$$

Thus, in this case, where one coalition is at capacity and the other coalition is not, if m, the initial number of coalitions producing the optimal quantity not at capacity, is less than the number found by entering appropriate values into the above formula, then the two coalitions should join.

(iii) Case 3, S_i and S_j both currently producing at capacity:

In this case, S_i and S_j are both currently producing at capacity and if they join together, will then be able to produce the optimal quantity not at capacity. Once again, from (12) and (13) derived earlier, the inequality that must be solved is:

(19) $$\frac{1}{m+1}\left(\left(h(S_i)+h(S_j)\right)\left(1-\frac{2H}{n}\right)\right) \le \frac{1}{((m+1)+1)^2}\left(\frac{n}{2}-2\left(H-\left(h(S_i)+h(S_j)\right)\right)+\frac{2}{n}\left(H-\left(h(S_i)+h(S_j)\right)\right)^2\right)$$

where $H = \sum_{t=m+1}^{r} h(S_t)$ from the original partition, before S_i and S_j join.

When solved for m, the above equation yields a parabola.[1] The m's that satisfy the profitability constraint are found in a unique range.[2] The bounds are:

(20) $$\frac{1}{\left(h(S_i)+h(S_j)\right)(n-2H)}\left((H+h(S_i)+h(S_j)-\tfrac{n}{2})^2 \pm \sqrt{(H-h(S_i)+h(S_j)-\tfrac{n}{2})^2\,(H+h(S_i)+h(S_j)-\tfrac{n}{2})^2}\right)$$

The lower bound of this range is always a negative number. However, the upper bound is positive. Thus, after the algebra (see appendix II.), the inequality to be examined is:

(21) $$m \le \frac{\left|(2\left(H+h(S_i)+h(S_j)\right)-n)(H-n)\right|}{2\left(h(S_i)+h(S_j)\right)(n-2H)}$$

The absolute value results from the square root of the squared quantities which must be positive. If n is large, this upper bound will often be positive.

It is worth noting that the case of S_i and S_j joining together and still not being able to produce the optimal quantity, which could occur, does not change anything. The optimal quantity for those other coalitions that could produce it in the beginning does not change.

$\pi(S_i \cup S_j; P) = \pi(S_i; P) + \pi(S_j; P)$. Therefore, it makes no difference if S_i and S_j do join in this scenario.

With the above results, given the number of players and the capacity of each of the players, an oligopoly can be be placed into partition function form. It should be noted at this point that if an oligopoly is looked at from the perspective of one, or a few, of the players and not all of the capacities are known, than cases can be found for the different capacity levels of the smaller firms where collusion would be profitable. Next a four player game will be examined.

V. An Example of a 4-player Game

First, to construct the utility, demand, cost, and profit functions, let:[3]

n=4, a=\$4.00, b=\$0.0006, c=\$1.00, $\Rightarrow$ A=\$5,000.00.

The individual capacities are: k_1=4,000, k_2=3,000, k_3=2,000, k_4=1,000, $\Rightarrow$ k_N=10,000.

Since $h_i = \frac{k_i}{A}$, the adjusted capacities are: $h_1 = \frac{4}{5}$, $h_2 = \frac{3}{5}$, $h_3 = \frac{2}{5}$, $h_4 = \frac{1}{5}$.

First, without capacity restrictions, $q_i^* = \frac{nA}{2(n+1)} = 2{,}000$.

Player 4 cannot produce this amount, so setting player 4 at capacity and recalculating yields:

$q_4 = k_4 = 1{,}000$, $q_1 = q_2 = q_3 = 2{,}250$.

Player 3 cannot produce this amount, so setting both players 3 & 4 at capacity yields:

$q_4 = k_4 = 1{,}000$, $q_3 = k_3 = 2{,}000$, $q_1 = q_2 = 2{,}333$, $q_N = 7{,}666$.

All players can produce these amounts, so the above is the non-cooperative partition solution.

The profits for the players are:

$\pi_1 = \pi_2 = \$1{,}633.33$, $\pi_3 = \$1{,}400$, $\pi_4 = \$700$, $\pi_N = \$5{,}366.66$.

For comparison, the monopoly solution is: $q_N^* = 5{,}000$, $\pi_N^* = \$7{,}500$.

At this point, coalitional formation questions can begin to be asked. First, should players 1 & 2 join, given they are both not at capacity and players 3 & 4 are separate and at capacity? Equation (18) finds that since players 1 & 2 are indeed the only two producing the optimal quantity not at capacity, i.e. $m = 2$, they should join. Indeed, checking finds: $q_{1\cup 2}^* = 3{,}500$, $\pi_{1\cup 2}^* = \$3{,}675 > \$3{,}266.66 = \pi_1 + \pi_2$ previously.

Next, the question of players 3 & 4 joining together will be addressed. First, players 1 & 2 are considered to be separate. Equation (21) states that 3 & 4 should join together when $m \leq \frac{34}{21} \approx 1.619$. Thus, they should not join. Indeed, one finds: $q^*_{3\cup 4} = q^*_1 = q^*_2 = 2{,}500$, $\pi^*_{3\cup 4} = \pi^*_1 = \pi^*_2 = \$1{,}875 < \$2{,}100 = \pi_3 + \pi_4$ previously. In examining the case of players 3 & 4 joining if players 1 & 2 have already joined, one finds that $q^*_{3\cup 4} = q^*_{1\cup 2} = 3{,}333$. Therefore, since players 3 & 4 cannot produce this quantity, they will both remain at capacity while $q^*_{1\cup 2} = 3{,}500$.

Now, going back to the non-cooperative partition, should player 4 join with either player 1 or 2? This case is where one player or coalition is at capacity and one player or coalition is not. Equation (18) finds that player 4 should only join a coalition producing the optimal quantity not at capacity if that coalition is the only one producing not at capacity. The exact result is $m \leq \frac{15}{14}$. Indeed, checking finds: $q^*_{2\cup 4} = q^*_1 = 2{,}666$, $q^*_3 = k_3 = 2{,}000$, $\pi^*_{2\cup 4} = \pi^*_1 = \$2{,}266.67 < \$2{,}333.33 = \pi_2 + \pi_4$ previously, $\pi^*_3 = \$1{,}600$. The results for player 4 joining with player 1 are similar.

The question of player 3 joining with either player 1 or 2 in the non-cooperative partition is very similar to the above case of player 4 joining with either player 1 or 2. Equation (18) states that player 3 should only join a coalition producing the optimal quantity not at capacity if that coalition is the only one producing not at capacity, or exactly $m \leq \frac{8}{7}$. Indeed checking yields: $q^*_{2\cup 3} = q^*_1 = 3{,}000$, $q^*_4 = k_4 = 1{,}000$, $\pi^*_{2\cup 3} = \pi^*_1 = \$2{,}733.33 < \$3{,}033.33 = \pi_2 + \pi_3$ previously, $\pi^*_4 = \$900$. The results for player 3 joining with player 1 are similar.

At this point, it is worth noting that beginning with the non-cooperative partition and examining which pairs of players could join, the possible partitions are $\{1,2,3,4\}$ or $\{1\cup 2,3,4\}$ only. Next, the partition $\{1\cup 2,3,4\}$ will be assumed to have formed, and further joining will be examined.

So assuming players 1 & 2 have joined, player 4 should join $1\cup 2$. Equation (18) suggests player 4 should join if $m \leq \frac{15}{14}$. The optimal values are then: $q^*_{1\cup 2\cup 4} = 4{,}000$, $q^*_3 = k_3 = 2{,}000$, $\pi^*_{1\cup 2\cup 4} = \$4{,}800 > \$4{,}725 = \pi^*_4 + \pi^*_{1\cup 2}$ previously, and $\pi^*_3 = \$2{,}400$.

Likewise, assuming players 1 & 2 have joined, player 3 should join $1\cup 2$. Equation (18) suggests player 3 should join if $m \leq \frac{8}{7}$. The optimal values are then: $q^*_{1\cup 2\cup 3} = 4{,}500$, $q^*_4 = 1{,}000$,

$\pi^*_{1\cup 2\cup 3} = \$6{,}075 > \$5{,}775 = \pi^*_3 + \pi^*_{1\cup 2}$ previously, and $\pi^*_4 = \$1{,}350$.

Now, assuming 1, 2, & 3 have joined, should player 4 join them? Equation (18) suggests player 4 should join if $m \leq \frac{19}{18}$, which is the case. Indeed, one finds: $q^*_{1\cup 2\cup 3\cup 4} = 5{,}000$, $\pi^*_{1\cup 2\cup 3\cup 4} = \$7{,}500 > \$7{,}425 = \pi^*_4 + \pi^*_{1\cup 2\cup 3}$ previously.

Likewise, assuming 1, 2, & 4 have joined, should player 3 join them? Equation (18) suggests player 3 should join if $m \leq \frac{9}{8}$, which is the case. Indeed, one finds: $q^*_{1\cup 2\cup 3\cup 4} = 5{,}000$, $\pi^*_{1\cup 2\cup 3\cup 4} = \$7{,}500 > \$7{,}200 = \pi^*_3 + \pi^*_{1\cup 2\cup 4}$ previously.

In summary, only five of the twelve possible partitions are found to have the possiblity of forming. The five partitions that are found to be possible are: $\{1,2,3,4\}$, $\{1\cup 2,3,4\}$, $\{1\cup 2\cup 3,4\}$, $\{1\cup 2\cup 4,3\}$, and $\{1\cup 2\cup 3\cup 4\}$.

Any value of the Shapley type works under the assumption that all coalitions, including the grand coalition, will form. There are values defined on partition function form games by R. Myerson [1977], E. Bolger [1987], and S. Merki [1991] which are axiomatically determined. However, in this model, as shown above, all coalitions may not form. Therefore, values that use the marginal contributions of players to coalitions are not applicable.

The division of profits is now left to be resolved by some other means than a value. Perhaps in cases such as a joint venture, division of profits could be done by a simple method. In this case, the profits could be divided by the percent invested by each member of the coalition. However, this solution does not reflect any other factors aside from a monetary sum, such as a brand name, brought to the coalition by each member. In a number of situations, this process would not be adequate.

division should be ad hoc rather than by a predetermined method?

VI. Economic Applications

The use of partition function form has several economic applications. Two of the most apparent are potential mergers and joint ventures. The results from this model could provide motivation for two firms producing a homogeneous product to merge or remain apart. The feasibility of the merger taking place can be examined first.

The area of joint ventures also provides a setting of applicability. Depending upon how the other members of the particular industry are colluding, analysis using this model will determine if a potential joint venture would be profitable.

The decision to enter into an industry is another application. The current environment can be examined. Then the decision to enter or stay out can be made. Further, the necessary production capacity can be determined, costs for this capacity can be estimated to provide for a good decision making tool.

John McDonald [1975], in his book The Game of Business, describes a business game comprised of American corporations deciding who would put up domestic communication satellites. Between 1960 and the mid 1970's, the idea of using satellites for domestic communication was developed to a point of being economically feasible. There were ten corporate groups involved. Some had the necessary technology to build the satellites, while others had the necessary "traffic", therefore there were potential gains from cooperation, even between firms that normally competed. Depending upon which corporations and the capacity sizes of the satellites to be placed in orbit, the coalitions have different worths. McDonald outlines the different worths for the various partitions that could form in his book. While in 1963, Lucas and Thrall were the first to develop the theory of partition function form games, McDonald's examination of the satellite game was the first application. Decisions for placing satellites in orbit with several players could be analyzed in the framework developed in this model.

VII. Summary and Conclusions

An oligopoly model has been created using the game-theoretic partition function form. Capacity restraints are placed upon the players with the assumption that all capacities are known. Based upon the idea of coalitions forming out of two smaller coalitions, using the axiom of superadditivity, sufficient conditions are derived for two coalitions joining together in each of the three possible scenarios: both not at capacity, one at capacity and one not at capacity, and both at capacity.

Footnotes

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[1]When $n > 2H$ the parabola is concave up, which is often the case. However, when $n < 2H$ the parabola is concave down.

[2]When $n < 2H$, then the m's that satisfy the profitability constraint are found in a strictly negative range. Thus, in other words, if $n < 2H$, then S_i and S_j should never join if they are both currently producing at capacity. Therefore, only the case of $n > 2H$ will be considered.

[3]The following assigned values are used by Martin Shubik in an example on page 127 in chapter five of his book *A Game-Theoretic Approach to Political Economy*, except in order to be consistent with the theory presented in this paper, the players have been reordered from smallest to largest capacity.

Appendix I.

The inequality that needs to be solved is:

$$\frac{1}{(m+1)^2}\left(\frac{n}{2}-2H+\frac{2}{n}H^2\right)+\frac{1}{m+1}\left(h(S_j)-\frac{2h(S_j)H}{n}\right)\leq\frac{1}{(m+1)^2}\left(\frac{n}{2}-2(H-h(S_j))+\frac{2}{n}(H-h(S_j))^2\right)$$

Where $H=\sum_{t=m+1}^{r} h(S_t)$ from the original partition, before S_i and S_j join.

$$\Rightarrow\left(\frac{n}{2}-2H+\frac{2}{n}H^2\right)+(m+1)\left(h(S_j)-\frac{2h(S_j)H}{n}\right)\leq\left(\frac{n}{2}-2(H-h(S_j))+\frac{2}{n}(H-h(S_j))^2\right)$$

$$\Rightarrow m\leq\left(\frac{\left(\frac{n}{2}-2(H-h(S_j))+\frac{2}{n}(H-h(S_j))^2\right)-\left(\frac{n}{2}-2H+\frac{2}{n}H^2\right)}{\left(h(S_j)-\frac{2h(S_j)H}{n}\right)}\right)-1$$

$$\Rightarrow m\leq\left(\frac{2-\frac{4H}{n}+\frac{2}{n}h(S_j)}{1-\frac{2H}{n}}\right)-1$$

$$\Rightarrow m\leq\frac{2n-4H+h(S_j)-n+2H}{n-2H}$$

$$\Rightarrow m\leq\frac{n-2H+h(S_j)}{n-2H}$$

Appendix II.

The equation that needs to be solved is when $m \leq$:

$$\frac{1}{\left(h(S_i)+h(S_j)\right)(n-2H)}\left(\left(H+h(S_i)+h(S_j)-\tfrac{n}{2}\right)^2+\sqrt{(H-h(S_i)+h(S_j)-\tfrac{n}{2})^2\ (H+h(S_i)+h(S_j)-\tfrac{n}{2})^2}\right)$$

$$\Rightarrow \quad m \leq \left(\frac{|H+h(S_i)+h(S_j)-\tfrac{n}{2}|}{\left(h(S_i)+h(S_j)\right)(n-2H)}\right)\left(|H+h(S_i)+h(S_j)-\tfrac{n}{2}| + |H-h(S_i)+h(S_j)-\tfrac{n}{2}|\right)$$

$$\Rightarrow \quad m \leq \left(\frac{|H+h(S_i)+h(S_j)-\tfrac{n}{2}|}{\left(h(S_i)+h(S_j)\right)(n-2H)}\right)\left(|H-n|\right)$$

$$\Rightarrow \quad m \leq \frac{\left(|2\left(H+h(S_i)+h(S_j)\right)-n|\right)\left(|H-n|\right)}{2\left(h(S_i)+h(S_j)\right)(n-2H)}$$

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