Properties of Simple Games

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Simple games are the class of cooperative games in characteristic function form for which $v(S)=1$ or $v(S)=0$ for all coalitions S of the set of players N. In words, every coalition is either "winning" (all-powerful) or "losing" (ineffectual). Simple games can be used to model the process of voting in parliamentary or other political bodies. When a motion is proposed, the coalition S that votes in favor of the proposal either carries enough weight under the existing voting scheme to enable the proposal to pass ($v(S)=1$) or it does not ($v(S)=0$).

Formally, a simple game is a pair (N, W) where $N=\{1,2,...,n\}$ is a set of players and W is the set of coalitions (subsets of N) which are winning. There are three conditions on W:

1. $N \in W$
2. $\emptyset \notin W$
3. If $T \in W$ and $S \supseteq T$ then $S \in W$.

Each condition is necessary if the simple game is to adequately model real-life voting: unanimity for or against a proposal is decisive; and adding players to a winning coalition (or removing them from a losing coalition) should not affect the outcome of a vote.

A simple game can be represented more compactly as (N, M) where M is the set of minimal winning coalitions. A minimal winning coalition is a winning coalition which contains no proper subset that is also winning. Thus the set M of minimal winning coalitions of a game does not contain two coalitions, one of which is a subset of the other. Such a set of sets is known as a clutter. One can easily derive W by appending players to each element of M. From now on, we will represent simple games with the (N, M) notation.

An interesting question to consider is: how many distinct sets of voting rules are possible for a voting body of n players? The set M uniquely determines a set of voting rules: for any coalition, it assigns the result "winning" or "losing". How many sets M are there on n players? To simplify matters, we only consider simple games of n players with no "dummies"; i.e., where each player has some say in the outcome of a vote. Thus every player must appear in at least one of the minimal winning coalitions.

When n=1, M={A} (where A is the only player) is the only possibility. When n=2, M={AB} or M = {A,B} are the two possibilities (M= {A} is not acceptable because B is a dummy). When n=3, M={A,B,C}, M= {AB,C}, M={AB,AC}, M={AB,AC,BC} and M={ABC} are the five simple games. Shapley [1] determined that there are also 20 4-player simple games, and proceeded to list all simple games of 4 or fewer players. He announced as well that there are 179 5-player simple games, which he later revised to 180. [3]

A paper-and-pencil algorithm for enumerating all n-player simple games follows. Let us impose an ordering within the sets M so that we may count them more efficiently. Within M, list coalitions in order of increasing size, and within size by alphabetical order. M is said to be in <u>reduced</u> <u>form</u> if no permutation of the players yields an alternate representation of M which preceeds M in the above ordering. Since we do not want to count duplicate sets of minimal winning coalitions, we must make sure that every set M is reduced before it is counted.

The algorithm is a tree. Off the origin node are the possible first coalitions {A}, {AB}, {ABC}, {ABCD} and {ABCDE}. Off node {A} are possible second coalitions {B}, {BC}, {BCD}, and {BCDE}, etc. Notice that we are culling out those coalitions whose addition would be identical to the ones mentioned above after a simple permutation of the players. For example, we needn't consider sets M beginning {A,B,...} and {A,C,...} since a swapping of players B and C (which would become necessary to put the latter set in reduced form) shows the two sets to be identical. Only nodes in which all players of N appear, and which is in reduced form are then counted. It is possible to add coalitions to a set in which all players already appear, so we must branch off all nodes wherever possible. Following is an illustration of the algorithm at work for the case n=5:

```
Origin Node
     A    AB   ABC  ABCD ABCDE
     {A,...}
          B    BC   BCD  BCDE
          {A,B,...}
               C    CD   CDE
               {A,B,C,...}
                    D    DE
                    {A,B,C,D,...}
                         E
                         {A,B,C,D,E}
                    {A,B,C,DE}
               {A,B,CD,...}
                    CE
                    {A,B,CD,CE}
                         DE
                         {A,B,CD,CE,DE}
               {A,B,CDE}
          {A,BC,...}
               BD   DE   BDE
               {A,BC,BD,...}
                    BE   CD   CE   CDE
                    {A,BC,BD,BE}
                         CD   CDE
                         {A,BC,BD,BE,CD}
                              CE
                              {A,BC,BD,BE,CD,CE}
                                   DE
                                   {A,BC,BD,BE,CD,CE,DE}
                         {A,BC,BD,BE,CDE}
```

```
                {A,BC,BD,CD,...}
                     CE   NOT IN REDUCED FORM
                {A,BC,BD,CE}
                     DE
                          {A,BC,BD,CE,DE}
                {A,BC,BD,CDE}
           {A,BC,DE}
           {A,BC,BDE}
                CDE
                {A,BC,BDE,CDE}
      {A,BCD,...}
           BCE
           {A,BCD,BCE}
                BDE
                {A,BCD,BCE,BDE}
                     CDE
                          {A,BCD,BCE,BDE,CDE}
      {A,BCDE}
```

These are the first twenty 5-player games. They can be placed in one-to-one correspondence with the twenty 4-player games on the players B,C,D and E by appending player A with "veto-power".

To describe the algorithm more precisely:

1. Begin at the origin node, where no coalitions have yet been specified. Think of each node as representing a (partial or complete) set of minimal winning coalitions in reduced form. The origin node represents the empty set.

2. From each node, trace all branches. A branch off a node is another (partial or complete) set of minimal winning coalitions identical to its parent except for an additional coalition. Since the coalitions of M are ordered, one need only consider those coalitions that are subsequent (equal-sized to the last coalition in M and alphabetically subsequent, or larger-sized) to those already in M.

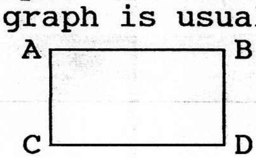
3. At each node M which contains all players, check to see if M is in reduced form. Determining whether M is in reduced form sometimes requires more than a casual glance. In theory, one can determine the reduced form of M by attempting all permutations of the players in M and choosing the resulting set whose ordering is smallest. In an n-player game this involves inspecting n! sets. However, the following method quickly resolves the issue for almost all five-player games. Using {AB,AC,BD,CD,ADE,BCE} as an example:

    1. Consider the coalitions of smallest size. Count the number of appearances for each player and list them from largest to smallest. For the example, this would be 2 2 2 2 0.

Begin listing possible two-player coalitions from alphabetically small to large -- AB uses up one A and one B, leaving
1 1 2 2 0. AC leaves 0 1 1 2 0. BC would come next, but would leave 0 0 0 2 0, which is unacceptable, because DD is not a coalition. So BD comes next instead, leaving the last two-player coalition to be CD.

Since {AB,AC,BC,BD} is indeed the subset of 2-player coalitions in our example, we know that so far it is in reduced form. If we had obtained a different subset of two-player coalitions, it would not necessarily have indicated a non-reduced form, however. Another test is required. (An example: M_1={ABC,ABD,ABE,ACD,CDE} and M_2={ABC,ABD,ACE,ADE,BCD} are distinctly different minimal winning sets, though they both have the player distribution 4-3-3-3-2, which by the technique just mentioned yields representation M_1. But in M_1 two players appear together in three coalitions, where the same is not true in M_2. So M_1 and M_2 are indeed distinct.)

2. Now proceed to the coalitions of next larger size, keeping in mind which players are "isomorphic" from preceeding sizes. After the coalitions of size two, players A, B, C, and D are all names for the same type of player - a player who is connected to two other players in a four player subset. For two player coalitions, a graph is usually helpful.



Player E, on the other hand, is an essentially different type of player - one who does not appear in any two-player winning coalition. Once the fundamental types of players have been distinguished, we use this information in examining the coalitions of next larger size. In our example, there are two three-player coalitions, each of which contains two players of "Type I" and player E. A little reflection reveals that {ADE,BCE} is indeed the best we can do lexicographically, since ACE and BDE are invalid because they contain winning proper subsets. Thus our example is indeed in reduced form.

If M is in reduced form, add it to the list of simple games already discovered. If not, disregard the rest of that branch.

4. Since the game under consideration is finite, both the number and the length of branches is finite. Once the entire tree is sketched, all sets M will have been discovered.

Note that in the illustration of the algorithm above, I did not test all branches from every node. For example, at {A,BC,BD,BE} I considered the branch which adds CD, but not those which add CE or DE for the simple reason that C,D, and E are different names for interchangeable elements of {A,BC,BD,BE}. A, being the only player who can win alone, has a distinct character. B is the only other player who can win in any two-player coalition. C,D, and E are all in the same position - needing the support of A or B to win. This and similar reasoning becomes important as the sets M grow in complexity.

Using this algorithm, it is possible to list the 180 5-player simple games. I owe thanks to David Housman, who finalized this list by writing a computer program to handle the 5-player case. The games are designated by their sets of minimal winning coalitions.

1	A,B,C,D,E
2	A,B,C,DE
3	A,B,CD,CE
4	A,B,CD,CE,DE
5	A,B,CDE
6	A,BC,BD,BE
7	A,BC,BD,BE,CD
8	A,BC,BD,BE,CD,CE
9	A,BC,BD,BE,CD,CE,DE
10	A,BC,BD,BE,CDE
11	A,BC,BD,CDE
12	A,BC,BD,CE
13	A,BC,BD,CE,DE
14	A,BC,BDE
15	A,BC,BDE,CDE
16	A,BC,DE
17	A,BCD,BCE
18	A,BCD,BCE,BDE
19	A,BCD,BCE,BDE,CDE
20	A,BCDE
21	AB,AC,AD,AE
22	AB,AC,AD,AE,BC
23	AB,AC,AD,AE,BC,BD
24	AB,AC,AD,AE,BC,BD,BE
25	AB,AC,AD,AE,BC,BD,BE,CD
26	AB,AC,AD,AE,BC,BD,BE,CD,CE
27	AB,AC,AD,AE,BC,BD,BE,CD,CE,DE
28	AB,AC,AD,AE,BC,BD,BE,CDE
29	AB,AC,AD,AE,BC,BD,CD

30	AB,AC,AD,AE,BC,BD,CE
31	AB,AC,AD,AE,BC,BD,CE,DE
32	AB,AC,AD,AE,BC,BD,CDE
33	AB,AC,AD,AE,BC,BDE
34	AB,AC,AD,AE,BC,BDE,CDE
35	AB,AC,AD,AE,BC,DE
36	AB,AC,AD,AE,BCD
37	AB,AC,AD,AE,BCD,BCE
38	AB,AC,AD,AE,BCD,BCE,BDE
39	AB,AC,AD,AE,BCD,BCE,BDE,CDE
40	AB,AC,AD,AE,BCDE
41	AB,AC,AD,BC,BD,CDE
42	AB,AC,AD,BC,BD,CE
43	AB,AC,AD,BC,BD,CE,DE
44	AB,AC,AD,BC,BDE
45	AB,AC,AD,BC,BDE,CDE
46	AB,AC,AD,BC,BE
47	AB,AC,AD,BC,BE,CDE
48	AB,AC,AD,BC,BE,DE
49	AB,AC,AD,BC,DE
50	AB,AC,AD,BCD,BCD
51	AB,AC,AD,BCD,BCE,BDE
52	AB,AC,AD,BCD,BCE,BDE,CDE
53	AB,AC,AD,BCDE
54	AB,AC,AD,BCE
55	AB,AC,AD,BCE,BDE
56	AB,AC,AD,BCE,BDE,CDE
57	AB,AC,AD,BE
58	AB,AC,AD,BE,BCD
59	AB,AC,AD,BE,BCD,CDE
60	AB,AC,AD,BE,CDE
61	AB,AC,AD,BE,CE
62	AB,AC,AD,BE,CE,BCD
63	AB,AC,AD,BE,CE,DE
64	AB,AC,AD,BE,CE,DE,BCD
65	AB,AC,ADE
66	AB,AC,ADE,BCD
67	AB,AC,ADE,BCD,BCE
68	AB,AC,ADE,BCD,BCE,BDE
69	AB,AC,ADE,BCD,BCE,BDE,CDE
70	AB,AC,ADE,BCD,BDE
71	AB,AC,ADE,BCD,BDE,CDE
72	AB,AC,ADE,BCDE
73	AB,AC,ADE,BDE
74	AB,AC,ADE,BDE,CDE
75	AB,AC,BC,ADE
76	AB,AC,BC,ADE,BDE
77	AB,AC,BC,ADE,BDE,CDE
78	AB,AC,BC,DE
79	AB,AC,BCD,BCE
80	AB,AC,BCD,BCE,BDE
81	AB,AC,BCD,BCE,BDE,CDE
82	AB,AC,BCD,BDE
83	AB,AC,BCD,BDE,CDE

84	AB,AC,BCDE
85	AB,AC,BD,ADE
86	AB,AC,BD,ADE,BCE
87	AB,AC,BD,ADE,BCE,CDE
88	AB,AC,BD,ADE,CDE
89	AB,AC,BD,CD,ADE
90	AB,AC,BD,CD,ADE,BCE
91	AB,AC,BD,CDE
92	AB,AC,BD,CE
93	AB,AC,BD,CE,ADE
94	AB,AC,BD,CE,DE
95	AB,AC,BDE
96	AB,AC,BDE,CDE
97	AB,AC,DE
98	AB,AC,DE,BCD
99	AB,AC,DE,BCD,BCE
100	AB,ACD,ACE
101	AB,ACD,ACE,ADE
102	AB,ACD,ACE,ADE,BCD
103	AB,ACD,ACE,ADE,BCD,BCE
104	AB,ACD,ACE,ADE,BCD,BCE,BDE
105	AB,ACD,ACE,ADE,BCD,BCE,BDE,CDE
106	AB,ACD,ACE,ADE,BCD,BCE,CDE
107	AB,ACD,ACE,ADE,BCD,CDE
108	AB,ACD,ACE,ADE,BCDE
109	AB,ACD,ACE,ADE,CDE
110	AB,ACD,ACE,BCD
111	AB,ACD,ACE,BCD,BCE
112	AB,ACD,ACE,BCD,BCE,CDE
113	AB,ACD,ACE,BCD,BDE
114	AB,ACD,ACE,BCD,BDE,CDE
115	AB,ACD,ACE,BCD,CDE
116	AB,ACD,ACE,BCDE
117	AB,ACD,ACE,BDE
118	AB,ACD,ACE,BDE,CDE
119	AB,ACD,ACE,CDE
120	AB,ACD,ACE,CDE
121	AB,ACD,BCDE
122	AB,ACD,BCE
123	AB,ACD,BCE,CDE
124	AB,ACD,CDE
125	AB,ACDE
126	AB,ACDE,BCDE
127	AB,CD,ACE
128	AB,CD,ACE,ADE
129	AB,CD,ACE,ADE,BCE
130	AB,CD,ACE,ADE,BCE,BDE
131	AB,CD,ACE,BDE
132	AB,CDE
133	ABC,ABD,ABE
134	ABC,ABD,ABE,ACD
135	ABC,ABD,ABE,ACD,ACE
136	ABC,ABD,ABE,ACD,ACE,ADE
137	ABC,ABD,ABE,ACD,ACE,ADE,BCD

138	ABC,ABD,ABE,ACD,ACE,ADE,BCD,BCE
139	ABC,ABD,ABE,ACD,ACE,ADE,BCD,BCE,BDE
140	ABC,ABD,ABE,ACD,ACE,ADE,BCD,BCE,BDE,CDE
141	ABC,ABD,ABE,ACD,ACE,ADE,BCDE
142	ABC,ABD,ABE,ACD,ACE,BCD
143	ABC,ABD,ABE,ACD,ACE,BCD,BCE
144	ABC,ABD,ABE,ACD,ACE,BCD,BDE
145	ABC,ABD,ABE,ACD,ACE,BCD,BDE,CDE
146	ABC,ABD,ABE,ACD,ACE,BCDE
147	ABC,ABD,ABE,ACD,ACE,BDE
148	ABC,ABD,ABE,ACD,ACE,BDE,CDE
149	ABC,ABD,ABE,ACD,BCD
150	ABC,ABD,ABE,ACD,BCD,CDE
151	ABC,ABD,ABE,ACD,BCDE
152	ABC,ABD,ABE,ACD,BCE
153	ABC,ABD,ABE,ACD,BCE,CDE
154	ABC,ABD,ABE,ACD,CDE
155	ABC,ABD,ABE,ACDE
156	ABC,ABD,ABE,ACDE,BCDE
157	ABC,ABD,ABE,CDE
158	ABC,ABD,ACD,BCD
159	ABC,ABD,ACD,BCE
160	ABC,ABD,ACD,BCE,BDE
161	ABC,ABD,ACD,BCE,BDE,CDE
162	ABC,ABD,ACD,BCDE
163	ABC,ABD,ACE,ADE
164	ABC,ABD,ACE,ADE,BCDE
165	ABC,ABD,ACE,BCDE
166	ABC,ABD,ACE,BDE
167	ABC,ABD,ACE,BDE,CDE
168	ABC,ABD,ACDE
169	ABC,ABD,ACDE,BCDE
170	ABC,ABD,CDE
171	ABC,ABDE
172	ABC,ABDE,ACDE
173	ABC,ABDE,ACDE,BCDE
174	ABC,ADE
175	ABC,ADE,BCDE
176	ABCD,ABCE
177	ABCD,ABCE,ABDE
178	ABCD,ABCE,ABDE,ACDE
179	ABCD,ABCE,ABDE,ACDE,BCDE
180	ABCDE

Power Indices

Intuitively, a power index measures the ability of a player in a voting body to influence a vote. The class of simple games known as weighted majority games provides a couple of examples. A weighted majority game is a simple game (N,W) where W is directly obtainable from the rule $[q;w_1, w_2,...w_n]$, where player i has number of votes w_i and q is the quota of votes needed to win; i.e.,

$$S\in W \text{ iff } \sum_{i\in S} w_i \geq q$$

Clearly, in an institution of majority rule such as the Supreme Court, each Justice has equal power, and any acceptable power index would be expected to reveal that. But in a majority weighted game such as [11;4,3,3,3,2,2,1,1,1] it is not immediately clear how much additional power the players with more votes have. Power indices were designed to answer these types of questions.

A power index, then, is a function from simple games on n players to R^n, which are the "power vectors", normalized so that their (nonnegative) components sum to 1. Following are the definitions of four power indices.

The Shapley-Shubik Index

This index considers the fractional percentage of the time that player i will be pivotal to the success of a winning coalition. All possible permutations of the players are considered, and in each permutation the players are considered to be joining the coalition in the order of their appearance. The player whose appearance first causes a losing coalition to become winning is the **pivotal** player for that permutation. The Shapley-Shubik index thus assigns to each player i a component of the power vector as follows:

$$\phi_i (N, M) = \frac{p_i}{n!}$$

where p_i is the number of times player i is pivotal.

The Bahnzaf Index

Consider all winning coalitions W of a simple game. Player i is critical to a coalition S⊆W if {S/{i}} is a losing coalition. The Bahnzaf index measures the fraction of time that player i is critical relative to the number of times all players are critical. Or,

$$\beta_i = \frac{\eta_i}{\sum_i^n \eta_i}$$

where η_i is the number of times player i is critical.

The Johnston Index

Akin to the Bahnzaf index, this index makes an additional distinction among vulnerable coalitions (coalitions in which one or more players are critical) according to the number of critical players in the coalition. Presumably, a player has more bargaining power if he is uniquely critical than if, for example, three other players are also critical. The Johnston index is defined as follows:

$$J_i = \frac{\sum_{i \in S \in V} \frac{1}{F(S)}}{|V|}$$

where F(S) is the number of critical players in coalition S and V is the set of all vulnerable coalitions.

The Deegan-Packel Index

This index is identical to the Johnston Index except that it takes into account only minimal winning coalitions. It is defined as follows:

$$\rho_i = \frac{1}{|M|} \sum_{S \in M_i} \frac{1}{|S|}$$

where M is the set of minimal winning coalitions, and M_i is the set of minimal winning coalitions containing player i. The idea here is that the larger the minimal winning coalition, the less power accrues to a player involved in it, because it is more difficult to form. The $1/|M|$ factor serves to normalize the power vector.

Calculation of Indices - Example

Consider the 5-player simple game with set of minimal winning coalitions M = {AB,CD,ACE,ADE,BCE}.

To calculate the Shapley-Shubik index for this game, list all permutations of the players. Assume for each permutation that the players join in the order from left to right. Underline the one player who causes the coalition to change from losing to winning. The first few of the 120 permutations are listed below:

ABCDE	ABCED	ABDCE	ABDEC	ABECD	ABEDC
ACBDE	ACBED	ACDBE	ACDEB	ACEBD	ACEDB

It turns out that player A is pivotal 32 times,
B is pivotal 22 times,
C is pivotal ~~30~~ times, 32
D is pivotal ~~24~~ times, 22
E is pivotal 12 times,
yielding a Shapley-Shubik index of [32,22,~~30~~ 32,~~24~~ 22,12]/120.

To calculate the Bahnzaf index, list all winning coalitions, underlining the players who are critical to the coalition's remaining winning:

AB	ABC	ABCD	ABCDE
CD	ABD	ABCE	
	ABE	ABDE	
	ACD	ACDE	
	ACE	BCDE	
	ADE		
	BCD		
	BCE		
	CDE		

Player A is critical 7 times,
B is critical 5 times,
C is critical 7 times,
D is critical 5 times,
E is critical 3 times,
so the Bahnzaf index for this game is [7,5,7,5,3]/27.

To calculate the Johnston index for this game, refer to the listing of winning coalitions above; but instead count fractional critical defections. It turns out that the Johnston index is

$$\left[\frac{11}{3},\frac{7}{3},\frac{11}{3},\frac{7}{3},1\right]/13$$

Finally, to calculate the Deegan-Packel index, consider only the set M and do the same calculation as for the Johnston index. This yields

$$\left[\frac{7}{6},\frac{5}{6},\frac{7}{6},\frac{5}{6},1\right]/5$$

Comparing the results from the four indices we have the following results, by percentage of power per player:

Index	A	B	C	D	E
Shapley-Shubik	26.7	18.3	~~25.0~~ 26.7	~~20.0~~ 18.3	10.0
Bahnzaf	25.9	18.5	25.9	18.5	11.1
Johnston	28.2	17.9	28.2	17.9	7.7
Deegan-Packel	23.3	16.7	23.3	16.7	20.0

Comparison of Indices - Oceanic Weighted Voting Game

It turns out that each of the four indices discussed embodies different assumptions about the voting process. For example, it is well known that the Shapley-Shubik index is more applicable to situations where the players have a good chance of convincing each other of their viewpoint, whereas the Bahnzaf index might better model a situation where the opposite is true. In yet another interpretation, it has been demonstrated that the Shapley-Shubik index adequately models voting situations in which the voters come into the vote with similar ideologies, whereas the Bahnzaf index better models votes where the players vote "heterogeneously". [2]

Each index also has different mathematical properties. An interesting case in point is how each index measures power in oceanic weighted voting games. These are majority weighted games in which alongside one or more "major players" exist a large (perhaps infinite) number of minor players of miniscule and equal power. An example is the game

$$[2;1,\frac{2}{n-1},\frac{2}{n-1},\ldots,\frac{2}{n-1}]$$

as n gets large. (There are (n-1) minor players, for a total of 3 votes in the game.) It seems intuitive that the major player controls a considerable part of the power (he holds 1/3 of the votes), though not all of it.

Applying each of the four indices described above yields the following results:

Shapley-Shubik Index

Within the set of all permutations of the n players, when n is odd the major player is pivotal if he joins the players in positions ((n-1)/2 + 1) through (n-1), and in positions (n/2 + 1) through (n-1) when n is even. Since the major player appears in each position equally often in the set of all permutations, the Shapley-Shubik index for the major player is as follows:

$$\phi_A = \begin{cases} \frac{n-1}{2n} & n\ odd \\ \frac{n-2}{2n} & n\ even \end{cases}$$

Clearly, as $n \mapsto \infty$ this index allots half the power to the major player, with the other half divided equally ~~between~~ among the minor players.

Bahnzaf Index

Here we count winning coalitions to which a player i is critical.

The major player is critical to all winning coalitions except the one containing all players and the one in which he is not involved (and all the minor players are). When n is odd, the number of winning coalitions to which the major player is critical is then:

$$\binom{(n-1)}{\frac{(n-1)}{2}} + \binom{(n-1)}{\frac{(n-1)}{2}+1} + \ldots + \binom{(n-1)}{(n-2)}$$

And the number of winning coalitions to which each minor player is critical is:

$$1 + \frac{1}{2} * \binom{n-1}{\frac{n-1}{2}}$$

Since there are (n-1) minor players, the Bahnzaf value for the major player is then:

$$\frac{\binom{n-1}{\frac{n-1}{2}} + \ldots + \binom{n-1}{n-2}}{n-1+\frac{n+1}{2}\binom{n-1}{\frac{n-1}{2}} + \ldots + \binom{n-1}{n-2}}$$

But as n gets large, this expression approaches 0, indicating that the major player holds none of the power! To see this, we will show that the third term in the denominator grows larger much more quickly than the numerator as a whole as n gets large, by making use of Sterling's factorial approximation:

$$n! \approx \sqrt{2\pi n}\left(\frac{n}{e}\right)^n$$

The middle binomial coefficient then becomes:

$$\binom{n-1}{\frac{n-1}{2}} \approx \frac{\sqrt{2\pi(n-1)}\left(\frac{n-1}{e}\right)^{n-1}}{2\pi\left(\frac{n-1}{2}\right)\left[\left(\frac{n-1}{2e}\right)^{\frac{n-1}{2}}\right]^2}$$

$$= 2^{n-1}\sqrt{\frac{2}{\pi(n-1)}}$$

Now, as n gets large, the third term of the denominator over the entire numerator goes to the following limit:

$$\lim_{n\to\infty} \frac{\binom{n+1}{2}2^{n-1}\sqrt{\dfrac{2}{\pi(n-1)}}}{\binom{n-1}{\frac{n-1}{2}}+\ldots+\binom{n-1}{n-2}}$$

$$\geq \lim_{n\to\infty} \frac{\binom{n+1}{2}2^{n-1}\sqrt{\dfrac{2}{\pi(n-1)}}}{2^{n-1}}$$

$$=\lim_{n\to\infty} \frac{n+1}{2}\sqrt{\frac{2}{\pi(n-1)}}$$

$$=\lim_{n\to\infty} \sqrt{\frac{(n+1)^2}{2\pi(n-1)}} = \infty$$

Thus the denominator grows much larger than the numerator as n gets large, and the Bahnzaf value for the major player does indeed approach 0.

Johnston Index

This is similar to the Bahnzaf index, except here we take into account the number of critical players in each winning coalition. In the coalition of all minor players without the major player, there are (n-1) critical defections. In the coalitions with (n-1)/2 minor players and the major player, there are (n-1)/2+1 critical defections. In all other winning coalitions except the one with all players, the only critical defection belongs to the major player. In the coalition of all players, there are no critical defections. The denominator in this index is the number of vulnerable coalitions, which is the number of winning coalitions minus the grand coalition. Thus, when n is odd (the situation is similar when n is even), the Johnston index for the major player is:

$$\frac{\dfrac{2}{n+1}\binom{n-1}{\frac{n-1}{2}}+1\left[\binom{n-1}{\frac{n-1}{2}+1}+\ldots+\binom{n-1}{n-2}\right]}{\binom{n-1}{n-1}+\binom{n-1}{\frac{n-1}{2}}+\ldots+\binom{n-1}{n-2}}$$

But we can ignore the first term of the numerator and the first two terms of the denominator as n gets large, because the middle binomial coefficient becomes insignificant compared to the sum of all binomial coefficients (or half of them) with increasing n. This is evident because:

$$\lim_{n\to\infty} \frac{2^{n-1}\sqrt{\dfrac{2}{\pi(n-1)}}}{2^{n-1}}$$

$$=\lim_{n\to\infty} \sqrt{\frac{2}{\pi(n-1)}} = 0$$

Thus, as n gets large, the limit of the Johnston index proves to be as counterintuitive as the Bahnzaf index in oceanic weighted majority games, though providing a different result: the major player is assigned all the power, rather than none.

Deegan-Packel Index

M, the set of minimal winning coalitions, consists of:

1. The set of all minor players
2. Every set including the major player and exactly (n-1)/2 minor players (if n is odd), or the major player and exactly n/2 minor players (if n is even).

Thus, when n is odd, applying the formula for the Deegan-Packel index to the major player:

$$\rho_A=\frac{1}{1+\binom{n-1}{\frac{n-1}{2}}}*\left[\binom{n-1}{\frac{n-1}{2}}*\frac{1}{\frac{n-1}{2}+1}\right]$$

$$=\frac{\binom{n-1}{\frac{n-1}{2}}}{\left[1+\binom{n-1}{\frac{n-1}{2}}\right]*\left[\frac{n-1}{2}+1\right]}$$

Now taking the limit as n goes to infinity yields the result that the major player has no power in large games. Only the Shapley index winds up giving the intuitive result for such games.

Characterization of Shapley Power Vectors

It may be possible to obtain all Shapley-Shubik power vectors for n-player games without enumerating all n-player games.

Since many simple games of n players have equivalent Shapley power vectors (excluding permutations on the n players), the number of these vectors, P(n), grows less quickly than the number of simple games on n players, S(n):

n	S(n)	P(n)
1	1	1
2	2	1
3	5	2
4	20	7
5	180	56

Listed below are the distinct Shapley-Shubik power vectors on simple games of up to 5 players with no dummies. Power vectors in n-player games with dummies are found by appending the appropriate number of zeroes to all power vectors of games with (n-1) and fewer players.

n=1

(1)

n=2

(1,1)/2

n=3

(2,2,2)/6
(4,1,1)/6

n=4

(3,3,3,3)/12
(5,5,1,1)/12
(6,2,2,2)/12
(7,3,1,1)/12
(4,4,2,2)/12
(5,3,3,1)/12
(9,1,1,1)/12

n=5

(48,3,3,3,3)/60
(42,7,7,2,2)/60
(39,9,4,4,4)/60
(37,12,7,2,2)/60
(36,6,6,6,6)/60
(34,9,9,4,4)/60
(33,18,3,3,3)/60
(33,8,8,8,3)/60
(32,12,7,7,2)/60
(32,7,7,7,7)/60
(30,15,5,5,5)/60
(30,10,10,5,5)/60
(29,9,9,9,4)/60
(28,13,13,3,3)/60
(28,13,8,8,3)/60

(28,8,8,8,8)/60
(27,27,2,2,2)/60
(27,12,12,7,2)/60
(27,12,7,7,7)/60
(26,11,11,6,6)/60
(25,15,10,5,5)/60
(24,14,14,4,4)/60
(24,14,9,9,4)/60
(24,9,9,9,9)/60
(23,23,8,3,3)/60
(23,18,8,8,3)/60
(22,17,17,2,2)/60
(22,17,12,7,2)/60
(22,17,7,7,7)/60
(22,12,12,7,7)/60
(21,21,6,6,6)/60
(21,16,11,6,6)/60
(21,11,11,11,6)/60
(20,20,10,5,5)/60
(20,15,10,10,5)/60
(20,10,10,10,10)/60
(19,19,14,4,4)/60
(19,19,9,9,4)/60
(19,14,14,9,4)/60
(19,14,9,9,9)/60
(18,18,18,3,3)/60
(18,18,8,8,8)/60
(18,13,13,13,3)/60
(18,13,13,8,8)/60
(17,17,12,12,2)/60
(17,17,12,7,7)/60
(17,12,12,12,7)/60
(16,16,16,6,6)/60
(16,16,11,11,6)/60
(16,11,11,11,11)/60
(15,15,15,10,5)/60
(15,15,10,10,10)/60
(14,14,14,14,4)/60
(14,14,14,9,9)/60
(13,13,13,13,8)/60
(12,12,12,12,12)/60

Notice that the least common denominator for the cases n=4 and n=5 are not n!, as in the formula for the Shapley value, but n!/2. This can be explained by the fact that when there are at least four players involved in the simple game, the number of times a player is pivotal

$$(n-s)!(s-1)!$$

is always even. For example, when C is pivotal in the player ordering ABCD it is also pivotal in BACD. Hence the least common denominator for Shapley-Shubik indices in 4-player games is 12, not 24. Continuing with this reasoning, the following serve as least common denominators for the next few larger sized simple games:

n	denominator
6	720/6 = 120
7	5040/6 = 720
8	40320/24 = 1440
9	362880/24 = 12560

As a first approach towards developing a method to find all possible Shapley-Shubik power vectors, we consider incremental strengthening of games. Starting with the (five-player) simple game M={ABCDE}, in which only the grand coalition is winning, consider those simple games in which exactly one additional coalition is winning. By monotonicity, this would have to be one of the games M={ABCD}, M={ABCE}, M={ABDE}, M={ACDE}, or M={BCDE}. By continuing to move through simple games in which one coalition at a time is changed from losing to winning, one can arrive at any simple game. Now, when a 4-player coalition is changed from losing to winning, the players in that coalition become critical, since the coalition is minimal winning. Thus the power index for these four players increases by (5-4)!(4-1)!/5! = 1/20. The lone player not in the newly winning coalition sees its power index decrease because it is no longer critical in the grand coalition. Thus its power decreases by (5-5)!(5-1)!/5! = 1/5.

By similar reasoning, when a three-player coalition changes from losing to winning, the players in that coalition increase in power by (5-3)!(3-1)!/5! = 1/30, while the players not in the coalition decrease by (5-4)!(4-1)!/5! = 1/20. All in all:

Size of Newly Winning Coalition	Change in Power Index of Coalition Members	Change in Power Index of Other Players
4	+ 1/20	- 1/5
3	+ 1/30	- 1/20
2	+ 1/20	- 1/30
1	+ 1/5	- 1/20

Thus, taking an example from 4-player simple games, any possible Shapley-Shubik power index must be of the form

$$\left[\frac{1}{4},\frac{1}{4},\frac{1}{4},\frac{1}{4}\right]+\sum_{i=1}^{4} a_i\left[\frac{1}{12},\frac{1}{12},\frac{1}{12},-\frac{1}{4}\right]+\sum_{j=1}^{6} b_j\left[\frac{1}{12},\frac{1}{12},-\frac{1}{12},-\frac{1}{12}\right]$$

$$+\sum_{k=1}^{4} c_k\left[\frac{1}{4},-\frac{1}{12},-\frac{1}{12},-\frac{1}{12}\right]$$

where the a_i, b_j, and c_k are nonnegative coefficients of all permutations of the last three vectors in the expression above. It turns out for the case n=4 that all nonnegative vectors of the above form are indeed Shapley-Shubik power vectors of 4-player simple games; however, for the n=5 case not all nonnegative vectors in the equivalent linear-combination expression (not derived here) are Shapley-Shubik power vectors. Thus it is not readily apparent how to obtain all Shapley-Shubik power vectors without calculating all n-player simple games.

References

1 Shapley, L.S. "Simple Games: An Outline of the Descriptive Theory," RAND Corporation Research Memorandum RM-1384, 1954.

2 Straffin, Jr., Philip D. "Power Indices in Politics," in Political and Related Models, edited by Steven J. Brams, William F. Lucas, and Philip D. Straffin, Jr. New York: Springer-Verlag, 1983.

3 Dubey, Pradeep and Shapley, Lloyd S., "Mathematical Properties of the Bahnzaf Index," Mathematics of Operations Research, Vol. 4, No. 2, May 1979.