SPANNING FOREST GAMES

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Cooperative Game Theory REU

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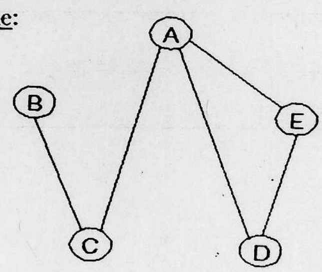
A *cooperative game* is a pair (N,w) where N is a set of *players* $\{1, 2,..., n\}$, and w is a real-valued function on the subsets of N, where $w(\emptyset) = 0$. The subsets of N, denoted by S, are called *coalitions.* The number $w(S)$ represents the value the individual players in S can obtain by cooperating as a group. A cooperative game is *superadditive* if $w(S \cup T) \geq w(S) + w(T)$ for all coalitions S and T satisfying $S \cap T = \emptyset$.

A *graph* is an ordered pair $G=(V,E)$, where V is a nonempty finite set of *vertices*, and E is a set of pairs of elements of V called *edges*; we sometimes denote V and E as $V(G)$ and $E(G)$, respectively [1]. A *subgraph of G* is any graph H for which $V(H) \subseteq V(G)$, and $E(H) \subseteq E(G)$ [1]. *The subgraph of G on* $S \subseteq V(G)$ is (S,E') where $E'=\{ \{a,b\} \in E(G): a,b \in S\}$. A graph G is *connected* if for every pair of vertices there is a *path* between the two vertices; in other words, there exists a way to get from one vertex to the other without crossing any vertex more than once; a graph is *disconnected* otherwise [1]. A *component* of a graph is a subgraph which is itself connected [1]. The *connectivity of G* corresponds to the minimum number of vertices which, when removed, disconnects G [1].

A *cycle* is a closed path, and a *tree* is a connected graph which has no cycles [1]. For a connected graph G, a *spanning tree, T,* is a tree subgraph of G where $V(G) = V(T)$ [1]. A *forest* is a graph which has no cycles; hence, if G is disconnected, then a forest consists of the spanning trees of each of the graph's components [1]. We define a *spanning forest, F,* of G to be a forest subgraph of G where $V(G) = V(F)$ and no additional edge can be added without creating a cycle.

A *graph game* is a game defined on a graph. We define one such game, a *spanning forest game,* to be an ordered pair (G,w) where $G = (V,E)$ is a graph, and w is a real-valued function on the subsets of V defined as follows: $w(S)$ is the number of edges in a spanning forest of the subgraph on S. In other words, $w(S) = |S| - c$, where c is the number of components of S.

Example of Spanning Forest Game:



Notation: w({A,B,C}) = w(ABC)

w(A) = w(B) = w(C) = w(D) = w(E) = 0,

w(AB) = 0, w(AC) = 1, w(AD) = 1, w(AE) = 1, w(BC) = 1,

w(BD) = 0, w(BE) = 0, w(CD) = 0, w(CE) = 0, w(DE) = 1,

w(ABC) = 2, w(ABD) = 1, w(ABE) = 1, w(ACD) = 2, w(ACE) = 2,

w(ADE) = 2, w(BCD) = 1, w(BCE) = 1, w(BDE) = 1, w(CDE) = 1,

w(ABCD) = 3, w(ABCE) = 3, w(ABDE) = 2, w(ACDE) = 3, w(BCDE) = 2,

w(ABCDE) = 4

An *allocation* for a cooperative game is a vector $x = (x_1, x_2, \ldots, x_n)$ where x_i is the *payoff* to the player i. An *allocation method* is a function which assigns an allocation to each cooperative game. An allocation method α is *efficient* if $\sum_{i=1}^{n} \alpha_i(N,w) = w(N)$; the allocation method α is *additive* if $\alpha(N,w+u) = \alpha(N,w) + \alpha(N,u)$ for all games (N,w) and (N,u); the allocation method α satisfies the property of *equal treatment* if $\alpha_i(N,w) = \alpha_j(N,w)$ for all games (N,w), where players $i,j \in N$ satisfy $w(S \cup \{i\}) = w(S \cup \{j\})$ for all $S \subseteq N - \{i,j\}$; the allocation method α satisfies the property of *no free lunch* if $\alpha_i(N,w)=0$ for all players $i \in N$ satisfying $w(S \cup \{i\}) = w(S)$ for all $S \subseteq N$. The *Shapley value* is the unique allocation method $\phi(N,w)$ satisfying the properties of efficiency, additivity, equal treatment, and no free lunch. The Shapley value can be found with the following formula:

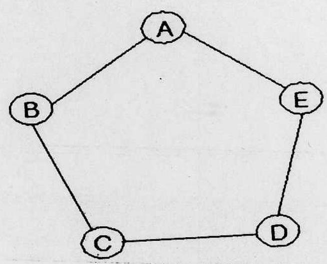
$$\phi_i(N,w) = \sum_{S \subseteq N} \frac{(s-1)!(n-s)!}{n!} [w(S) - w(S - \{i\})], \text{ where } s=|S| \text{ and } n=|N|. \text{ [2]}$$

One of our goals is to find a method for determining the Shapley value of a player using only the information found in the graph, rather than using the formula mentioned above. We can do this for some spanning forest games using the following theorems.

<u>Theorem 1</u>: In a cycle with n vertices, the Shapley value for any vertex is $1 - \frac{1}{n}$.

<u>Proof</u>: The proof follows directly from equal treatment and efficiency.

<u>Example Using Theorem 1</u>:



$$\phi_A = \phi_B = \phi_C = \phi_D = \phi_E = \frac{4}{5}$$

<u>Theorem 2</u>: If the graph $G = (N,E)$ associated with a spanning forest game (G,w) is a tree, then for each $i \in N$, $\phi_i(N,w) = \frac{\deg(i)}{2}$.

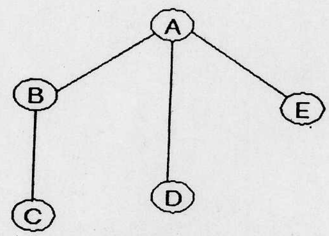
<u>Proof</u>: Let $G=(N,E)$ be a tree. Define $w^e(S)$ on $e \in E$ and $S \subseteq N$ as follows:

$$w^e(S) = \begin{cases} 1 & \text{if both vertices incident to } e \text{ are in } S \\ 0 & \text{otherwise} \end{cases}$$

$\sum_{e \in E} w^e(S)$ is equal to the number of edges in G for which both incident vertices are in S; since G is a tree, $\sum_{e \in E} w^e(S)$ is equal to the number of edges in a spanning forest on S. Thus, $\sum_{e \in E} w^e(S) = w(S)$, by the definition of a spanning forest game. Thus, $\phi(w) = \phi(\sum_{e \in E} w^e)$. Furthermore, by additivity, $\phi(\sum_{e \in E} w^e) = \sum_{e \in E} \phi(w^e)$, so $\phi(w) = \sum_{e \in E} \phi(w^e)$. It follows that for each $i \in N$, $\phi_i(w) = \sum_{e \in E} \phi_i(w^e)$.

We consider a player $i \in N$. For an edge e incident to i, $\phi_i(w^e) = \frac{1}{2}$ by efficiency, equal treatment, and no free lunch. Thus, $\phi_i(w) = \sum_{e \in E} \phi_i(w^e) =$ (number of edges incident to i)$*\frac{1}{2} = \frac{\deg(i)}{2}$.

<u>Example</u> <u>Using</u> <u>Theorem</u> <u>2</u>:



$\phi_A = \frac{3}{2}$, $\phi_B = 1$, $\phi_C = \phi_D = \phi_E = \frac{1}{2}$

<u>Theorem</u> <u>3</u>: Suppose $G = (V,E)$ is a graph and $\{E_1,E_2,...,E_k\}$ is a partition of E which partitions cycles (if C is a cycle in G, then $C \subseteq E_j$ for some $j = 1,2,...,k$). Let $G_j = (V,E_j)$ for $j = 1,2,...,k$. Then $\phi(W_G) = \sum_{j=1}^{k} \phi(W_{G_j})$.

<u>Proof</u>:

Let F_j be a spanning forest for G_j.

Then by the definition of forest size games, $W_{G_j}(S) = |F_j|$.

Let $F = F_1 \cup ... \cup F_k$

So F is a subset of E.

Suppose F has a cycle. Then the cycle would be in some F_j because $\{E_1,...,E_k\}$ partitions cycles. But this contradicts the fact that F_j is a forest. So, F has no cycles.

Suppose F does not span G. Then there exists an edge e which can be added to F without creating a cycle. But this edge would have to be added to some F_j without creating a cycle in G_j, which contradicts the fact that F_j is a spanning forest. Therefore, F spans G.

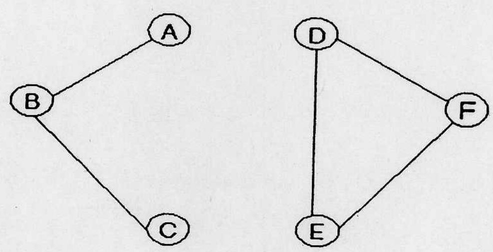
Hence, F is a spanning forest for G.

So $W_G(S) = |F| = \sum_{j=1}^{k} |F_j| = \sum_{j=1}^{k} W_{G_j}(S)$.

Thus $\phi(W_G) = \phi\left(\sum_{j=1}^{k} W_{G_j}\right)$.

And by additivity, $\phi(W_G) = \sum_{j=1}^{k} \phi(W_{G_j})$.

Example Using Theorem 3:



$\phi_A = \phi_C = \frac{1}{2}$, $\phi_B = 1$, $\phi_D = \phi_E = \phi_F = \frac{2}{3}$

Theorem 4: Suppose $G=(V,E)$ is a graph in which all cycles are edge disjoint.

Then $\phi_i(W_G) = \frac{\deg_G(i)}{2} - \sum_{C \in \mathcal{C}(G,i)} \frac{1}{|C|}$, where $\mathcal{C}(G,i) = \{C \subseteq E(G) : C \text{ is a cycle containing } i\}$.

Proof:

Let $E = E_0 \cup E_1 \cup \ldots \cup E_k$, where $E_1,\ldots,E_k$ are cycles in G and $E_0 = E - (E_1 \cup \ldots \cup E_k)$ is the remaining forest. Note that $\{E_0, E_1,\ldots, E_k\}$ is a partition of E since cycles are edge disjoint.

Let $G_j = (V,E_j)$.

By Theorem 1, for $j = 1,\ldots,k$,

$$\phi_i(W_{G_j}) = \begin{cases} 1 - \dfrac{1}{|E(G_j)|} & \text{if } i \text{ is adjacent to } E(G_j) \\ 0 & \text{otherwise} \end{cases}$$

In other words, $$\phi_i(W_{G_j}) = \begin{cases} \dfrac{\deg_{G_j}(i)}{2} - \dfrac{1}{|E(G_j)|} & \text{if } i \text{ is adjacent to } E(G_j) \\ \dfrac{\deg_{G_j}(i)}{2} & \text{otherwise} \end{cases}$$

And, by Theorem 2, for $j = 0$,
$$\phi_i(W_{G_0}) = \frac{\deg_{G_0}(i)}{2}.$$

$\{E_0, E_1, \ldots, E_k\}$ partitions cycles, so by Theorem 3,

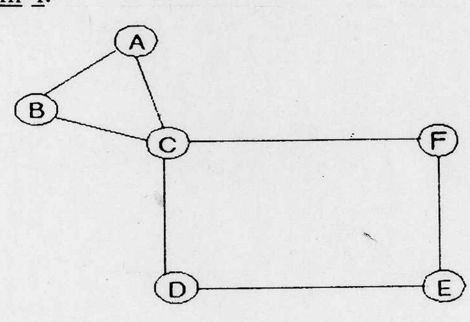
$$\phi_i(W_G) = \sum \phi_i(W_{G_j})$$

$$= \frac{\deg_{G_0}(i)}{2} + \sum_{j=1}^{k} \frac{\deg_{G_j}(i)}{2} - \sum_{j^*} \frac{1}{|E(G_j)|}$$

$$= \frac{\deg_G(i)}{2} - \sum_{C \in \mathcal{C}(G,i)} \frac{1}{|C|}$$ because $\{E_0, \ldots, E_k\}$ is a partition of E.

$*$ where i is adjacent to $E(G_j)$

<u>Example Using Theorem 4</u>:



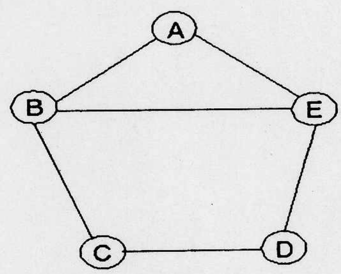
$\phi_A = \phi_B = \frac{2}{3}$, $\phi_C = \frac{17}{12}$, $\phi_D = \phi_E = \phi_F = \frac{3}{4}$.

<u>Theorem 5</u>: Suppose $G=(V,E)$ is a graph and $E_1, E_2 \subseteq E$ for which $E_1 \cup E_2 = E$, $E_1 \cap E_2 = \{e\}$ for

$e \in E$, and any cycle $C \subset E$ for which $e \in C$ must satisfy $C \subseteq E_1$ or $C \subseteq E_2$. Let $G_i = (V, E_i)$ for $i=1,2$. If $i \in V$ is not incident to e, then $\phi_i(G,w) = \phi_i(G_1,w) + \phi_i(G_2,w)$. If $i \in V$ is incident to e, then $\phi_i(G,w) = \phi_i(G_1,w) + \phi_i(G_2,w) - \frac{1}{2}$.

For a proof of Theorem 5, see Darren Lim's paper.

Example Using Theorem 5:



$\phi_A = \frac{2}{3}, \phi_B = \frac{11}{12}, \phi_C = \frac{3}{4}, \phi_D = \frac{3}{4}, \phi_E = \frac{11}{12}$

Another allocation method to consider is the *Tau Value*[3], which represents a compromise between the maximum and minimum amounts to which each player may relistically be entitled. For a player $i \in N$, the maximum and minimum entitlements, respectively, are given by:

$M_i = w(N) - w(N - \{i\})$, and

$m_i = \max\{w(S) - \sum_{j \in S - \{i\}} M_j : i \in S \subseteq N\}$

The Tau Value is the efficient compromise between the maximum and minimum entitlements and can be found with the following formula:

$\tau_i(N,w) = \lambda m_i + (1 - \lambda) M_i$

where $\lambda = \dfrac{\sum_{i \in N} M_i - w(N)}{\sum_{i \in N} M_i - \sum_{i \in N} m_i}$.

In a graph G, we say that a vertex v is a *cut vertex* if its removal disconnects a component of G [4]. Using the cut vertices of a graph, we can determine the Tau value for each player in the corresponding spanning forest game using the following three theorems:

Theorem 6: If a vertex v_s is an isolated vertex in the graph $G=(N,E)$ associated with a spanning forest game (G,w), then $\tau_{v_s}(N,w) = 0$.

Proof: For an isolated vertex v_s, $M_{v_s} = w(N) - w(N - \{v_s\}) = 0$.

By the definition of a spanning forest game, $w(S) \leq |S| - 1$. For any $i \in N$, there exists a subset $S \subseteq N$ for which $w(S) = |S| - 1$, this being $S = \{i\}$, which maximizes $w(S)$. Also, since $M_i \geq 1$ for all $i \in N$, $\sum_{j \in S - \{i\}} M_j \geq |S| - 1$. Thus, for any $i \in N$ there exists an $S \subseteq N$ for which $\sum_{j \in S - \{i\}} M_j = |S| - 1$, this being $S = \{i\}$, which minimizes $\sum_{j \in S - \{i\}} M_j$. It follows that for any $i \in N$, $\max\{w(S) - \sum_{j \in S - \{i\}} M_j : i \in S \subseteq N\} = |S| - 1 - (|S| - 1) = 0$. Thus, for any $i \in N$, $m_i = 0$.

Since $M_{v_s} = 0$ and $m_{v_s} = 0$, $\tau_{v_s} = \lambda(0) + (1 - \lambda)0 = 0$.

Because of the results of Theorem 6, we need no longer consider isolated vertices when calculating the Tau value. Hence, in the following two theorems, we assume that the graph G contains no isolated vertices, ie. that all isolated players have been removed from the game.

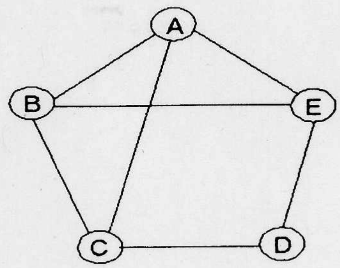
Theorem 7: If the graph $G=(N,E)$ associated with a spanning forest game (G,w) contains no cut vertices, then $\tau_i(N,w) = \frac{w(N)}{n}$ for all $i \in N$, where $n=|N|$.

Proof: Let i be any vertex in N. Since i is not a cut vertex, $N - \{i\}$ is connected, so $M_i = w(N) - w(N - \{i\}) = 1$.

For any $S \subseteq N$, $\sum_{j \in S - \{i\}} M_j = |S| - 1$. Furthermore, for any $i \in N$ there exists a subset $S \subseteq N$ which is connected, this being $S=\{$players in the component containing $i\}$. Thus, an S exists for which $w(S)=|S| - 1$. In this case, $w(S) - \sum_{j \in S - \{i\}} M_j = |S| - 1 - (|S| - 1) = 0$; this is the maximal case because if $S \subseteq N$ is disconnected, then $w(S) < |S| - 1$, so $w(S) - \sum_{j \in S - \{i\}} M_j < 0$. Thus, $m_i = \max\{w(S) - \sum_{j \in S - \{i\}} M_j : i \in S \subseteq N\} = 0$.

Since $M_i=1$ and $m_i=0$ for every $i \in N$, $\lambda = \frac{n - w(N)}{n}$. Thus, $\tau_i(N,w) = \frac{n - \mathrm{w}(N)}{n} m_i + \left(1 - \frac{n - \mathrm{w}(N)}{n}\right) M_i = 0 + \frac{n - (n - w(\text{“}N))}{n} = \frac{w(N)}{n}$ for all $i \in N$.

Example using Theorem 7:



$\tau_A = \tau_B = \tau_C = \tau_D = \tau_E = \frac{4}{5}$

Theorem 8: If v_c is a cut vertex on the graph $G=(N,E)$ associated with a spanning forest game (G,w), and its removal separates the component containing it into k components, then $\tau_{v_c} = k * \tau_{v_n}$, where v_n is a vertex in G which is not a cut vertex.

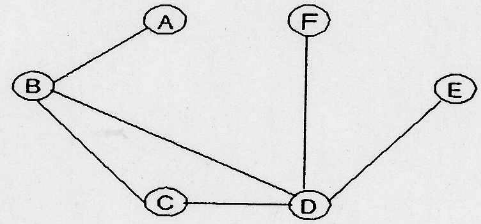
Proof: G contains a non-cut vertex v_n by a theorem given by Behzhad and Chartrand[4] which states that any graph for which $|V| \geq 2$ must contain at least two non-cut vertices.

Since v_n is not a cut vertex, $M_{v_n} = w(N) - w(N - \{v_n\}) = 1$. Also, since v_c is a cut vertex whose removal divides the component containing it into k components, $M_{v_c} = w(N) - w(N - \{v_c\}) = w(N) - [w(N) - k] = k$.

By the definition of a spanning forest game, $w(S) \leq |S| - 1$. For any $i \in N$, there exists a subset $S \subseteq N$ for which $w(S) = |S| - 1$, this being $S = \{i\}$, which maximizes $w(S)$. Also, since $M_i \geq 1$ for all $i \in N$, $\sum_{j \in S - \{i\}} M_j \leq |S| - 1$. Thus, for any $i \in N$ there exists an $S \subseteq N$ for which $\sum_{j \in S - \{i\}} M_j = |S| - 1$, this being $S = \{i\}$, which minimizes $\sum_{j \in S - \{i\}} M_j$. It follows that for any $i \in N$, $\max\{w(S) - \sum_{j \in S - \{i\}} M_j : i \in S \subseteq N\} = |S| - 1 - (|S| - 1) = 0$. Thus, for any $i \in N$, $m_i = 0$.

For the non-cut vertex v_n, $\tau_{v_n}(N,w) = \lambda(0) + (1 - \lambda)1 = 1 - \lambda$. Furthermore, for the cut vertex v_c, $\tau_{v_c}(N,w) = \lambda(0) + (1 - \lambda)k = k(1 - \lambda)$. Clearly, $\tau_{v_c} = k * \tau_{v_c}$.

Example Using Theorem 8:



Let $x = \tau_i$ for every cut vertex i:

$\tau_A = x$, $\tau_B = 2x$, $\tau_C = x$, $\tau_D = 3x$, $\tau_E = x$, $\tau_F = x$.

$w(N) = 5$.

By efficiency, $x + 2x + x + 3x + x + x = w(N)$, so $9x = 5$, and $x = \frac{5}{9}$.

Thus, $\tau(G,w) = (\frac{5}{9}, \frac{10}{9}, \frac{5}{9}, \frac{15}{9}, \frac{5}{9}, \frac{5}{9})$.

It is important to note that the definition for the Tau value given above only applies to games which are *quasi-balanced*; these are games for which $m_i \leq M_i$ for all $i \in N$, and $\sum_{i \in N} m_i \leq w(N) \leq \sum_{i \in N} M_i$. However, the two proofs given above show that for any spanning forest game, $m_i = 0$ and $M_i \leq 1$ for all $i \in N$, so $m_i \leq M_i$ for all $i \in N$. Furthermore, $\sum_{i \in N} m_i = 0$ and $\sum_{i \in N} M_j \leq |N|$. Since $0 \leq w(N) \leq |N| - 1$, it follows that $\sum_{i \in N} m_i \leq w(N) \leq \sum_{i \in N} M_i$. Thus, any spanning forest game is quasi-balanced.

Our future work may include considering other allocations with respect to spanning forest games. We may also determine special allocations for graph games in general, or attempt to characterize games which may be represented by graphs.

REFERENCES

1. Gould, Ronald. Graph Theory. The Benjamin/Cummings Publishing Company, Inc: Menlo Park, California (1988).

2. Roth, Alvin E., ed., The Shapley Value, Cambridge University Press: Cambridge (1988).

3. Tijs, S.H. (1981). bounds for the core and the τ-value. In: Game Theory and Mathematical Economics (O. Moeschlin and D. Pallaschke eds.). North-Holland Publishing Company, Amsterdam, 123-132.

4. Behzad, Mehdi and Chartrand, Gary. Introduction to the Theory of Graphs. Allyn and Bacon Inc: Boston (1971).