Reasonableness, Monotonicity and Rationality

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Abstract: Reasonableness, Monotonicity, and Rationality

This paper ~~will~~ examines different cost allocation methods for superadditive cooperative games in regard to the properties reasonableness, monotonicity, and rationality. Because on 3-person games a special condition concerning reasonableness holds, the group of games with four or more players is examined more intensively, to better represent the general class of superadditive n-person games. My research covers the three properties with respect to the Shapley Value, Nucleolus, Tau Value, Banzhaf Value, and the Nucleolus of the Anticore Dual Game.

Introduction

Consider a game (N,v), where N = (1, 2, 3,...,n) is the set of players and v is a real-valued function defined on all coalitions $S \subseteq N$ satisfying $v(\emptyset) = 0$. The vector $\mathbf{x} = \{x_1, x_2, \ldots, x_n\}$ gives each player i an allocation x_i. We assume throughout this paper that the games are superadditive.

Player i's marginal contribution to coalition S is defined as $v(S) - v(S - \{i\})$. John Milnor (1952) proposed that a **reasonable** payoff allocation x_i, for player i, should be no more than i's maximum marginal contribution.

Rationality insures that each player, either individually or as part of a coalition, will receive at least their stated worth. To accept less than one started with would be "irrational."

Monotonicity insures that the allocation a player receives fluctuates accordingly if a coalition containing that player increases or decreases in value. Certainly a player whose payoff has decreased while their stated worth has increased would not be satisfied.

Definitions

A game (N,v) satisfies **superadditivity** if $v(S \cup T) \geq v(S) + v(T)$ $\forall$ S and T satisfying $S \cap T = \emptyset$.

Convex?

The set of imputations $I = \{x \in R^n : \sum_{i \in N} x_i = v(N)$ and $x_i \geq v(i)$ for all $i \in N\}$
This set is also known as the **individually rational set.**

The **group rational** set, also known as the **core, C.**

$$C = \{x \in R^n : \sum_{i \in N} x_i = v(n) \text{ and } \sum_{i \in S} x_i \geq v(S) \ \forall \ i \in N\}.$$

A payoff vector $b = (b_i : i \in N)$ is called **individually reasonable** iff it satisfies $b_i \leq \max_{S \ni i} (v(S) - v(S - \{i\}))$ $\forall$ $i \in N$
The set of all b_i's is denoted **B.**

The **anticore, A,** is defined as the set of all imputations that are **group reasonable** such that for every S,

$$\sum_{S \ni i} x_i \leq \max_{T \supseteq S} [v(T) - v(T - S)].$$

An allocation procedure φ, is ind. ratl, grp ratl, ind. reas., group reason. at a game (N,v) if $\varphi(N,v)$ satisfies the respective property.

An allocation procedure φ is **aggregate monotonic** if for any games (N,v) and (N,w),

$$v(N) \geq w(N) \text{ and } v(S) = w(S) \ \forall \ S \subset N \Rightarrow \varphi_i(v) \geq \varphi_i(w) \ \forall \ i \in N.$$

An allocation procedure φ is **group monotonic** if for any games (N,v) and (N,w),
$v(T) \geq w(T)$ for some $T \subset N$ and $v(S) = w(S)$ $\forall$ $S \neq T$, $S \subset N \Rightarrow \varphi_i(v) \geq \varphi_i(w)$ for all $i \in T$.

need point on class definitions

The **Shapley Value** for player i is the average of the marginal values player i brings to the coalition of all players over all possible individual orderings. More specifically,

$$\varphi_i(N,v) = \sum_{S \subseteq N} \frac{(s-1)!(n-s)!}{n!} [v(S) - v(S - \{i\})]$$

where $s = |S|$, and n is the number of players in the game.

The **Banzhaf Value** for player i is, after zero normalization, proportional to the sum of the marginal values player i brings to each possible coalition.

$$\beta_i(N,v) = v(i) + \frac{\sum_{i \in S \subseteq N} [v(S) - v(S - \{i\}) - v(i)]}{\sum_{j \in N} \sum_{j \in S \subseteq N} [v(S) - v(S - \{j\}) - v(j)]} [v(N) - \sum_{j \in N} v(j)]$$

The **Nucleolus** is the imputation that minimizes $e(x)$

lexicographically.
$e(x,S) = v(s) - \sum_{i \in s} x_i$ is the excess of group S relative to the cost allocation x, and $e(x)$ is the vector of $e(x,S)$ ordered from largest to smallest.

The **Tau Value** gives player i a compromise payment between a maximum and a minimum, determined by separable values. The maximum payoff to an individual is no more than his or her separable value, so we define M as follows:

$$M_i = v(N) - v(N-\{i\}).$$

The minimum payoff player i should accept is the grand coalition less all the other players' separable values, defined as:

$$\mu_i = \max \{v(S) - \sum_{j \in S-\{i\}} M_j : i \in S \subseteq N\}.$$

The **Tau Value** on quasibalanced games is defined as:

$$\tau_i(N,v) = \lambda \mu_i + (1-\lambda) M_i$$

where

$$\lambda = \frac{\sum_{i=1}^{n} M_i - v(N)}{\sum_{i=1}^{n} M_i - \sum_{i=1}^{n} \mu_i}.$$

The **Anticore Dual Game** is defined as $(N,\hat{v})$ where $\hat{v}(S) = \max [v(T)-v(T-S)]$ for all $S \subseteq T \subseteq N$ as given by the original (N,v) game.

	Shapley Value	Nucleolus	Tau Value	Banzhaf Value	Negative Anticore Dual Game Nucleolus
Indiv. Rat'l	all SA games	all SA games	all SA quasi. games	all SA games	with grp reas. alloc. all SA games?
Group Rat'l	convex games; not all SA	all ~~SA~~ games with grp. ratn alloc.	convex games?	convex games? not all SA	convex games, elsewhere ?
Indiv. Reas.	all SA games	all SA games	all SA quasi. games	all SA games?	all SA games
Group Reas.	n=3 SA games, n=4 SA games, convex games not all with grp. reas. alloc.	n=3 SA games, convex games, balanced games	n=3 SA qu.games, convex ? qu.games?	convex games? not all with grp. reas. alloc.	all ~~SA~~ games with grp. reas. alloc.
Agg. Mono.	all SA games	3 player ? not all ...	convex quasi. games?	all SA games?	?
Group Mono.	all SA games	not in SA or convex games	convex quasi. games	all SA games ?	?

Table 1. Where do certain values have the properties for superadditive (SA) games?

balanced = with grp ratn alloc.

1. Three Person Games

For three person superadditive games, an allocation that is individually rational (i.e., in the set of imputations **I**) and individually reasonable is also group reasonable. The anticore **A** equals the intersection of **B** with **I** (Shubik, 1982). Therefore, the following values are group reasonable for n=3: Shapley Value, nucleolus, per capita nucleolus (Jew, 1989), the Tau Value, and possibly the Banzhaf Value.

Theorem 1: $\mathbf{A} = \mathbf{B} \cap \mathbf{I}$ $\forall$ 3-person superadditive games.

Proof:

Step 1.(A) Must show $\mathbf{A} \subseteq \mathbf{B}$.

The members of **A** that are eligible for membership in **B** are the singeltons {i}'s.

$$x_i \leq \max_{T \supseteq \{i\}} [v(T) - v(T-\{i\})] = b_i \Rightarrow x_i \leq b_i \Rightarrow \mathbf{A} \subseteq \mathbf{B}.$$

(B) Must show $\mathbf{A} \subseteq \mathbf{I}$.

Let $S = \{jk\}$, $x \in \mathbf{A}$.

We know $x_j + x_k \leq \max [v(jk)-v(\emptyset), v(ijk)-v(i)]$,

$\leq v(N)-v(i)$ by superadditivity,

$x_j + x_k = v(N)-x_i$ by efficiency, so

$v(N) - x_i \leq v(N) - v(i)$,

$x_i \geq v(i) \Rightarrow x \in \mathbf{I} \Rightarrow \mathbf{A} \subseteq \mathbf{I}$.

$\mathbf{A} \subseteq \mathbf{B}$ and $\mathbf{A} \subseteq \mathbf{I} \Rightarrow \mathbf{A} \subseteq \mathbf{B} \cap \mathbf{I}$.

Step 2. Must show $\mathbf{B} \cap \mathbf{I} \subseteq \mathbf{A}$.

(A) Singletons. Look at $x_i \in \mathbf{B} \cap \mathbf{I}$.

$v(i) \leq x_i \leq b_i = \max [v(S) - v(S-\{i\})]$

then certainly $x_i \leq \max_{T \ni S} [v(T) - v(T-S)]$, where $S=\{i\}$,

$\Rightarrow x \in \mathbf{A}$.

(B) Pairs. Look at $x_i, x_j \in \mathbf{B} \cap \mathbf{I}$:

$x_i + x_j = v(N) - x_k$ by efficiency,

$\leq v(N) - v(k)$ by individual rationality,

$= \max_{T \ni \{ij\}} [v(T) - v(T-\{ij\})]$ by superadditivity,

$\Rightarrow x_i + x_j \in \mathbf{A}$.

$\Rightarrow \mathbf{B} \cap \mathbf{I} \subseteq \mathbf{A}$.

$\mathbf{B} \cap \mathbf{I} \subseteq \mathbf{A}$ and $\mathbf{A} \subseteq \mathbf{B} \cap \mathbf{I} \Rightarrow \mathbf{B} \cap \mathbf{I} = \mathbf{A}$. □

Because the Shapley Value, Nucleolus, Per Capita Nucleolus, and Tau Value are individually rational and individually reasonable, they are group reasonable and contained in the ~~in the~~ anticore for three person games. While I know that the Banzhaf Value is individually rational, I have not been able to prove individual reasonablenss, so I can only propose that the Banzhaf Value is group reasonable for three-person games.

Step 1 of the previous proof will extend equally well to other n-person games. This means that Group Reasonableness implies individual reasonableness and rationality and that the anticore, when nonempty, is contained within the intersection of B (individually reasonable) with I (individually rational) for all n-person games.

2. Games With Four or More Players

2.1 Rationality

The Shapley Value is individually rational by definition:

$$\sum_{S\subseteq N} \frac{(s-1)!(n-s)!}{n!} = 1 \text{ and } v(S)-v(S-\{i\}) \geq v(i) \text{ for all } S\subseteq N.$$

The Shapley Value is not in the core **C** for all superadditive games, but is always group rational for convex games.

By definition, the Nucleolus is individually rational. By examining the similarities between the definitions of the Nucleolus and a nonempty core, it can be shown that the Nucleolus is group rational (Jew, 1989).

The Banzhaf Value is individually rational by definition. The Banzhaf Value is not group rational for all superadditive games, but I believe the value is in the core for convex games. I have not yet been able to prove or disprove my proposal, but I intend to continue trying.

Show example (veto power game)

For all superadditive quasibalanced games, the Tau Value is shown to individually rational by definition in a proof by Curiel (1988), which I will not reprint here. As with the Banzhaf Value, a sampling of random quasibalanced convex games has led me to believe that the Tau Value is in the core for convex games. Again, I have yet to prove or find a counterexample to my proposal.

This argument works when $A(N,v) \neq \emptyset$, but does not work when $A(N,v) = \emptyset$. There is a proof that is similar to Jew's proof that the nucleolus is ind. reas.

The negative Nucleolus of the Anticore Dual Game $[-\mathrm{V}(N,-\hat{v})]$ is individually rational in the original (N,v) game, because the negative nucleolus is in the anticore of the original (N,v) game (to be proved later) and by Theorem 1, $\mathbf{A}(N,v) \subseteq \mathbf{I}(N,v)$. For convex games, the negative nucleolus is in the core of the original game, because **C**=**A** for convex games (Housman, 1990).

Theorem 2: **C**=**A** for convex games.

Proof:

Lemma: If v is convex, then

$$v(N)-v(N-S) = \max\ [v(T)-v(T-S) : T\supseteq S].$$

Proof of Lemma: Suppose $S \subseteq T \subseteq N$. Then $T \cup (N-S) = N$ and $T \cap (N-S) = T-S$. By convexity,

$$v(T) + v(N-S) \le v(T \cup (N-S)) + v(T \cap (N-S)) = v(N) + v(T-S).$$

So, $v(T) - v(T-S) \le v(N) - v(N-S)$.

$$\begin{aligned} \text{Now, } x \in A(v) &\Leftrightarrow x(S) \le v(N) - v(N-S) \text{ for all } S \subseteq N \\ &\Leftrightarrow v(N-S) \le v(N) - x(S) \text{ for all } S \subseteq N \\ &\Leftrightarrow v(N-S) \le x(N-S) \text{ for all } S \subseteq N \\ &\Leftrightarrow x \in C(v). \square \end{aligned}$$

2.2 Reasonableness

By definition, the Shapley Value is individually reasonable for all superadditive games. The Shapley Value is group reasonable for three person games by Theorem 1, and four person games by the extension of Theorem 1 and the fact that

$$\begin{aligned} \phi_i + \phi_j &= (1/6)[v(N)-v(kl)] + (1/6)[v(N)-v(jkl)+v(j)] \\ &+ (1/6)[v(N)-v(ikl)+v(i)] + (1/6)[v(ijl)-v(l)] \\ &+ (1/6)[v(ijk)-v(k)] + (1/6)[v(ij)] \\ &\le \max\,[v(T) - v(T-\{ij\}) : T \supseteq \{ij\}]. \end{aligned}$$

The Shapley Value is also group reasonable for all convex games.
Jew (1989) presents a 5 player game having group reasonable allocations but the Shapley Value is not one.

By Jew (1989), the Nucleolus is individually reasonable because it is lexicographically minimal. By Theorem 1, the nucleolus is group reasonable for three player games, and is in the anticore for convex balanced games otherwise, since for convex balanced games the core (which equals the anticore) is nonempty, and the nucleolus is always contained in the core.

The complexity of the Banzhaf Value makes its analysis exceedingly difficult. By examining randomly obtained superadditive and convex games, I can propose (but not prove) that the Value is individually reasonable for all superadditive games and group reasonable for all convex games. The Banzhaf Value is not in the anticore for all nonconvex games, as is shown by the four-player game below:

$v(1)=v(2)=v(3)=v(4)=0$ $v(12)=.405$ $v(13)=.494$ $v(23)=.250$
$v(14)=.462$ $v(24)=.256$ $v(34)=.086$ $v(123)=.583$
$v(124)=.680$ $v(134)=.960$ $v(234)=.379$ $v(N)=1$, where
$\beta = \{.392, .169, .212, .227\}$.

This is grp. reas.

The Tau value is always individually reasonable, by a simple proof (Housman, 1990).

Proof:
By definition of M_i, $M_i \le b_i$, and because the game is quasibalanced, $\mu_i \le b_i$. Thus $\tau_i(N,v) = \lambda\mu_i + (1-\lambda)M_i \le \lambda b_i + (1-\lambda)b_i = b_i$. $\square$

I suspect that the Tau Value is also group reasonable, as I have not been able to find a counterexample. But this still needs to be proved. I have not yet looked at non-quasibalanced games.

The negative Nucleolus of the Anticore Dual Game is, by definition, individually and group reasonable in the original (N,v) for all superadditive games.

2.3 Monotonicity

The Shapley Value is well known to be both aggregate and group monotonic (Young, 1985) for all superadditive games.

The Nucleolus is not uniformly aggregate monotonic over the class of all superadditive games (Jew, 1989). The convex zero-normalized game $v(12)=.1$, $v(13)=.15$, $v(23)=v(24)=.2$, $v(14)=.14$, $v(34)=.23$, $v(123)=v(134)=v(234)=.45$, $v(124)=.65$, $v(N)=1$ gives the nucleolus $\nu = \{.249, .274, .175, .303\}$. Upon decreasing $v(124)$ to .5 (the game retains convexity) and $\nu = \{.25, .25, .25, .25\}$, thus the nucleolus is not group monotone for even convex games.

I believe, but have not yet proved, that the Banzhaf value is group monotone. Because the partial derivative of the Banzhaf value with respect to v(N) is shown to be nondecreasing, the value is proved to be aggregate monotone for all superadditive games (Léotard, 1990). I would have liked to see this proof written up.

The Tau value fails tests of aggregate and group monotonicity for many superadditive quasibalanced nonconvex games, but I believe the value is aggregate monotonic on convex games. example? Because the partial derivative of the Tau Value for player i (for a convex game) with respect to coalitions containing i is shown to be nondecreasing, the value is proved to be group monotonic for all convex quasibalanced games. These sentences are inconsistent where are the proofs?

I have not yet examined the negative nucleolus of the Anticore Dual Game; I believe that the value will react towards monotonicity in a similar manner as the regular nucleolus of the original (N,v) game.

3. Values with Two Properties

An allocation procedure satisfying more than one of the group reasonableness, group rationality, and group monotonicity properties would certainly be more agreeable to the players of the game.

3.1 Group Rational and Group Monotone

Young (1985) and Jew (1988) show that there exists no allocation method that is both group rational and group monotone on games of $|N| \geq 4$. For convex games, however, the Shapley Value is both group rational and group monotone. The Tau and Banzhaf values are also potentially both group rational and group monotone for convex games.

3.2 Group Rational and Group Reasonable

Jew (1989) shows that the Shapley Value and nucleolus are not both group rational and reasonable consistently over all superadditive games. On convex games, however, the Shapley value and nucleolus are both group rational and reasonable. The Tau value is not both group rational and group reasonable over all n-person games, but may have both properties in convex quasibalanced games. I have also found three and four person games with nonempty cores where the Banzhaf value is group reasonable but not group rational, thus, like the Tau value, this value does not have both properties consistently over all n-person cooperative games, but may have over the class of convex games. The negative nucleolus of the Anticore Dual Game has both properties over all convex games and wherever else the allocation is group rational for the original (N,v) game.

3.3 Group Reasonable and Group Monotone

Jew (1989) showed that the nucleolus is not both group reasonable and group monotone. The convex game described earlier to show that the nucleolus is not group monotone proves that the allocation can never have both properties for even convex games. The Shapley Value has both properties for all superadditive three and four player games, and for convex games otherwise. I believe, but need to prove, that the Tau and Banzhaf values possess both properties as well for convex games.

4. Suggestions for Future Study

1. Complete the chart in Figure 1, and extend to other values. Create linear programs to find any possible counterexamples.

2. Characterize games for which $A=B\cap I$. $A\neq B\cap I$ over the class of all superadditive four-player games, but perhaps has equality when certain restriction are added.

3. Characterize games for which $A\neq\emptyset$. Look at the Anticore Dual Game: in general, members of the negative core of the (N,-v) game will be members of the anticore of the normal (N,v) game (Housman, 1990). In this manner we can tell when the anticore of a normal game is nonempty. Also, because the Shapley value is group reasonable in the class of all superadditive four player games, the anticore is nonempty there.

4. Examine monotonicity properties of the negative nucleolus of the Anticore Dual Game.

5. Axiomatically characterize the anticore as a solution concept.

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Dear Jane,

I hope that you have been able to relax after eight weeks of mathematics. Unfortunately, I have had administrative work for the Council on Undergraduate Research, lectures for the New Jersey Governor's School, and software library development for Introductory Statistics. Jeanne and I took a few hours off for our anniversary, but now I'm back finishing up REU stuff.

I have enclosed (1) a request for an evaluation of the program, (2) the original of your REU report upon which I have written a number of suggestions, (3) a copy of participants' whereabouts, and (4) a copy of Lori Jew's thesis. Please return your evaluation to me by September 17. If you would like a revised copy of your report to be sent to the NSF and other interested persons, please return your revision by September 17. I have held off sending out copies of student reports pending each student's decision whether or not to revise. There is no requirement to revise your report; it is up to you based upon your time and interest. You requested only reports that might be necessary or helpful. In late September, I will send you a copy of my monotonicity paper. If there are any articles I discuss below which you do not have, I can also send you copies. Just let me know.

You undertook a very open-ended area and did a good job of organizing the morass. In addition, you obtained three nice and original results: the Shapley value is group reasonable on four player games, the Banzhaf value is aggregate monotone on superadditive games, and the nucleolus is not group monotone on all convex games (in fact, your example can be used to show that the nucleolus is not even aggregrate monotone on all convex games). Unfortunately, you did not really have sufficient time to complete the organization of the morass and so your report suffers in several substantial ways. First, incorrect proofs are given for the individual rationality of the negative anticore dual game nucleolus and for the Banzhaf value not being group reasonable on all games having group reasonable allocations. Second, proofs are missing for the aggregate monotonicity of the Banzhaf value and the group (or is it aggregate?) monotonicity of the tau value on convex games. Third, counterexamples are missing that show the Banzhaf value is not group rational on some games containing group rational allocations and show the tau value is not aggregate monotone on a nonconvex quasibalanced game. Fourth, there should be definitions of the following format for each of the six properties of interest: a value θ has property P at the game (N, v) if condition C holds. The rational and reasonable definitions you have relate to allocations not values, and the monotonicity definitions are for the class of all games rather than a specific game or subclass. Fifth, I do not think that the section on values having two properties adds any further information to the report.

Having had a few weeks to let the dust settle, I now would not use the term "dual," because the dual of a dual is not the original game although the dual of a dual usually is the original object when the term "dual" is used. I will devote the next few paragraphs showing you how I would develop this material now.

Definition. Suppose (N, v) is a cooperative game. The group reasonable cover of v is the function $\hat{v}$ defined by $\hat{v}(S) = \max \{ v(T) - v(T \setminus S) : S \subseteq T \}$ for all $S \subseteq N$.

Proposition. Suppose (N, v) is a cooperative game, and $\hat{v}$ is the group reasonable cover of v. If v is superadditive, then $-\hat{v}$ is superadditive. Further, x is individually [resp., group] reasonable in (N, v) if and only if $-x$ is individually [resp., group] rational in $(N, -\hat{v})$.

Proof. Suppose v is superadditive and $R \cap S = \emptyset$. Let T be a superset of $R \cup S$ satisfying $\hat{v}(R \cup S) = v(T) - v(T \setminus R \cup S)$. Then $\hat{v}(R \cup S) = v(T) - v(T/S) + v(T/S) - v(T \setminus R \cup S) \leq \hat{v}(S) + \hat{v}(R)$ which implies that $-\hat{v}$ is superadditive. The second conclusion follows from the observations that $\hat{v}(N) = v(N)$ and $x(S) \leq \hat{v}(S)$ if and only if $-x(S) \geq -\hat{v}(S)$. The notation x(S) means $\sum_{i \in S} x_i$.

Definition. The group reasonable cover prenucleolus, denoted $\hat{\nu}$, of the game (N, v) is the efficient allocation that minimizes $\hat{e}(x)$ lexicographically where $\hat{e}(x, S) = x(S) - \hat{v}(S)$ and $\hat{e}(x)$ is the vector of $\hat{e}(x, S)$, $S \subseteq N$ ordered from largest to smallest.

Note that the grc prenucleolus of the game (N, v) is simply the negative of the prenucleolus of the game $(N, -\hat{v})$. Now the prenucleolus is individually rational on superadditive games and group rational on games with group rational allocations. Hence, the Proposition implies that the grc prenucleolus is individually reasonable on superadditive games and group reasonable on games with group reasonable allocations. It is somewhat trickier to show that the grc prenucleolus is individually rational; however, the proof is similar to the standard rationality proofs for the prenucleolus.

The prenucleolus is individually rational on the superadditive game (N, v). Indeed, suppose $x_i < v(i)$ for some $i \in N$. Then for any coalition S containing i, it follows that $e(x, S) = v(S)$

$- x(S) \geq v(S-i) + v(i) - x(S) > v(S-i) - x(S-i) = e(x, S-i)$. Consider the allocation y defined by $y_i = x_i + (n-1)\epsilon$ and $y_j = x_j + \epsilon$ for $j \neq i$ for an $\epsilon > 0$. Now ϵ can be chosen small enough so that $e(x, S) > e(y, S) > e(y, S-i) > e(x, S-i)$ for all coalitions S containing i. Hence, $e(y)$ is lexicographically smaller than $e(x)$. Thus, x is not the prenucleolus.

The prenucleolus y is group rational on the game (N, v) which has at least one group rational allocation. Suppose x is a group rational allocation for (N, v). Then $x(S) \geq v(S)$ for all coalitions S. This implies that $e(x, S) = v(S) - x(S) \leq 0$ for all coalitions S. Since the prenucleolus minimizes the maximum excess, $e(y, S) \leq 0$ for all coalitions S which implies that $y(S) \geq v(S)$ for all coalitions S. Thus, the prenucleolus is group rational.

The grc prenucleolus is individually rational on the game (N,v). Indeed, suppose $x_i < v(i)$ for some $i \in N$. Then for any coalition S containing i, I claim (and prove below) that $\hat{e}(x, S) < \hat{e}(x, S-i)$. Consider the allocation y defined by $y_i = x_i + (n-1)\epsilon$ and $y_j = x_j + \epsilon$ for $j \neq i$ for an $\epsilon > 0$. Now ϵ can be chosen small enough so that $e(x, S) < e(y, S) < e(y, S-i) < e(x, S-i)$ for all coalitions S containing i. Hence, $e(y)$ is lexicographically smaller than $e(x)$. Thus, x is not the grc prenucleolus. I now prove the claim. Suppose that S is a coalition that contains i. I am to show that $x(S) - \hat{v}(S) < x(S-i) - \hat{v}(S-i)$. Since $x_i < v(i)$, it is sufficient to show that $\hat{v}(S) \geq \hat{v}(S-i) + v(i)$. Let T be a superset of $S-i$ satisfying $\hat{v}(S-i) = v(T) - v(T\backslash(S-i))$. If $i \in T$, then $\hat{v}(S) \geq v(T) - v(T\backslash S)$ and $v(T\backslash(S-i)) \geq v(T\backslash S) + v(i)$ which together yield the desired inequality. If $i \notin T$, then $\hat{v}(S) \geq v(T+i) - v((T+i)\backslash S) \geq v(T) + v(i) - v(T\backslash(S-i))$ which is the desired inequality.

Finally, I have a few comments on future study. A completion of the table would involve characterizing the classes of games on which each value satisfies each property. This is a very hard project with several inelegant characterization likely. An interesting partial completion of the table is to ask for each value-property pair whether the value satisfies the property on all games within the following classes: superadditive, with group reasonable allocations, with group rational allocations, convex, and n-player for various n. This would make a good undergraduate thesis. In this connection, you should read several chapters in Theo Driessen, *Cooperative Games, Solutions and Applications.* A nice characterization of the games with group reasonable allocations (nonempty anticore) has more potential for publication. I would first consider "size dependent" games, i.e., v(S) depends only on the size of S. I think that the general case would require an understanding of the characterization of games with nonempty core (a proof can be found in G. Owen, *Game Theory*). Let me know what you plan to pursue.

I will close with a few words about recommendations. I would be happy to write you a recommendation for graduate study or employment upon request. It is my policy to always share with you a copy of my letter of recommendation for you. If there is sufficient time between your request and the receipt deadline, I will send you a first draft for comment. You received two good letters of recommendation when you applied to the REU program. David Shannon's was better for the REU program.

Good luck digesting all of this!

Sincerely,

David Housman