A Characterization of the Extreme Monotonic Extensions of a Partially Defined Game

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When only limited information exists about the worths of certain subsets of individuals in a game, standard methods cannot compute payoffs. One solution is to allocate as dictated by some specific extension of the game. The extreme points of monotonic extensions are characterized.

1 Background

A *cooperative game* is a pair (N, w) where $N = \{1, 2, \cdots, n\}$ is a set of *players* and $w: 2^N \to \mathbb{R}$ with $w(\emptyset) = 0$ gives the worth obtainable by the cooperation of each subset of players. A *value* associates with each (N, w) a vector in $\mathbb{R}^n$ representing the payoff to each player. A game is *monotonic* if $w(S) \leq w(T)$ whenever $S \subseteq T$.

Letscher (1990) defines a *partially defined cooperative game (PDG)* to be a triple (N, Ω, v) where $\Omega \subseteq 2^N$ are the coalitions whose worths are known and $v: \Omega \to \mathbb{R}$ gives these worths. We require $\emptyset, N \in \Omega$. An *extension* of this PDG is a game (N, w) with $v(S) = w(S)$ for all $S \in \Omega$. We define a *partial extension* of this PDG to be a PDG $(N, \bar{\Omega}, \bar{v})$ where $\Omega \subseteq \bar{\Omega}$ and $v(S) = \bar{v}(S)$ for all $S \in \Omega$. Define this PDG to be *monotonic* if $v(S) \leq v(T)$ for all $S, T \in \Omega$ with $S \subseteq T$.

The object of Ventrudo and Wallman (1991) is to determine value on all PDGs $P = (N, \Omega, v)$. One approach is to find the set $\mathfrak{M}(P)$ of all monotonic extensions of the game, select some "central" point in this set, and apply some value to this game. A geometric characterization of $\mathfrak{M}(P)$ facilitates the selection of such a point.

View a game (N, w) as a vector in $\mathbb{R}^{2^n}$. It is easy to verify that $\mathfrak{M}(P)$ is a bounded convex set. An *extreme point* of a convex set C is an $x \in C$ such that if $c_1 + c_2 = 2x$ for some $c_1, c_2 \in C$, then $c_1 = c_2 = x$. We characterize ex($\mathfrak{M}(P)$), the extreme points of $\mathfrak{M}(P)$.

2 A Condition Sufficient for Extremity

Index each factor in the product $\{0,1\}^b$, where $b = 2^n - |\Omega|$, by a different element of $2^N \backslash \Omega$. For any $\alpha \in \{0,1\}^b$, any monotonic PDG $P = (N, \Omega, v)$, define the game (N, v^α) as follows. Arrange the elements of $2^N \backslash \Omega$ in order of nondecreasing cardinality: $S_0, S_1, \cdots, S_b$. Define $v^\alpha(S) = v(S)$ for all $S \in \Omega$. Assume that $v^\alpha(S_i)$ has been defined for all $i < l$. Define

$$v^\alpha(S_l) = \begin{cases} \max\{v^\alpha(S) \mid S \subseteq S_l \text{ and } (S \in \Omega \text{ or } S = S_i \text{ for some } i < l)\} & \text{if } \alpha(S_l) = 0 \\ \min\{v^\alpha(S) \mid S \supseteq S_l,\ S \in \Omega\} & \text{if } \alpha(S_l) = 1 \end{cases}$$

Theorem: If $(N, v^\alpha) = (N, w)$ for some $\alpha \in \{0,1\}^b$, then $(N, w) \in \mathrm{ex}(\mathfrak{M}(P))$.

Proof: Ventrudo and Wallman (1991). □

We claim the converse fails. Define P by $N = \{12345\}$, $\Omega = \{S \subseteq N \mid |S| = 0, 1, 4, \text{ or } 5\}$,

$$v(12345) = 2,\ v(i) = 0,$$

$$v(2345) = 1,\ \text{otherwise } v(ijkl) = 2.$$

Define the extension (N, w) by $w(123) = w(234) = w(235) = w(23) = w(12) = 1$;

$$w(124) = w(125) = 2;\ \text{for other } |S| = 2 \text{ or } 3,\ w(S) = 0.$$

which is monotonic. To show w extreme, suppose $\exists \Delta \in \mathbb{R}^{2^n}$ such that $w \pm \Delta$ are monotonic. Then

$$w(23) = w(234) = w(235) = v(2345) \Rightarrow \Delta(23) = \Delta(231) = \Delta(235) = 0$$

$$w(123) = w(23) \Rightarrow \Delta(123) = 0$$

$$w(12) = w(123) \Rightarrow \Delta(12) = 0$$

Clearly for all other $S \subseteq N$, $\Delta(S) = 0$, making Δ the zero vector. So w is extreme.

However there is no α such that $v^\alpha = w$, because $v^\alpha(12)$ can only be 0 or 2, never 1.

3 A Necessary and Sufficient Condition

Theorem: Consider any monotonic PDG $P_0 = \left(N, \Omega_{P_0}, v\right)$. Arrange the elements of $\{v(S) \mid S \in \Omega\}$ in increasing order $0 = a_1 < a_2 < \cdots < a_k$. Define families of PDGs $\mathfrak{P}_0, \mathfrak{P}_1, \cdots, \mathfrak{P}_k$ inductively.

Base step: $\mathfrak{P}_0 = \{P_0\}$.

Inductive steps:

(X1) If $P \in \mathfrak{P}_{i-1}$ then $\widehat{P} \in \mathfrak{P}_i$ where

$$\Omega_{\widehat{P}} = \Omega_P \cup \bigcup_{v(S) = a_i} 2^S$$

$$v_{\widehat{P}}(T) = a_i \qquad \text{for all } T \in \bigcup_{v(S) = a_i} 2^S \backslash \Omega_P$$

$$v_{\widehat{P}}(T) = v_P(T) \qquad \text{for all } T \in \Omega_P.$$

(X2) If $P \in \mathfrak{P}_i$ and $\exists T^* \in 2^N \backslash \Omega_P$ such that $\max\{v_P(S) \mid S \subset T^*, S \in \Omega_P\}$ exists and equals a_i, then $\bar{P} \in \mathfrak{P}_i$, where

$$\Omega_{\bar{P}} = \Omega_P \cup 2^{T^*}$$

$$v_{\bar{P}}(T) = a_i \qquad \text{for all } T \in 2^{T^*} \backslash \Omega_P$$

$$v_{\bar{P}}(T) = v_P(T) \qquad \text{for all } T \in \Omega_P.$$

We claim that $\mathfrak{P}_k$ is the set of extreme points $ex(\mathfrak{M}(P_0))$.

Proof: Establish both inclusions.

Claim 1: $\mathfrak{P}_k \subseteq \text{ex}(\mathfrak{M}(P_0))$.

Proof: Since $\bigcup_{v(S) = a_k} 2^S = 2^N$, we have (X1) $\Rightarrow$ every $P \in \mathfrak{P}_k$ is totally defined. We claim that for $i = 0, \cdots, k$, every $P \in \mathfrak{P}_i$ is

(1) a monotonic partial extension of P_0

satisfying the following:

(2) If $R \in \Omega_P$ and $v_P(R) \leq a_i$ then $2^R \subseteq \Omega_P$.

(3) For all $T \in \Omega_P, \forall \Delta \in \mathbb{R}^{2^n}$ with $\Delta(T) \neq 0$, we have that one of $v_P \pm \Delta$ is not a monotonic partial extension of P_0.

The result would then follow from the case $i = k$. Induct on i. The claim holds for $i = 0$. Assume it holds for $i < l$ and prove it holds for $i = l$.

Consider any $\widehat{P}$ that arises via (X1) from some $P \in \mathcal{P}_{l-1}$, and any $T \in \bigcup_{v(S)=a_l} 2^S \backslash \Omega_P$. First show (1) holds for $\widehat{P}$. For all $A \subseteq T$ with $A \in \Omega_P$, we have P monotonic $\Rightarrow v_P(A) \leq v_P(S) = a_l$. For all $A \supseteq T$ with $A \in \Omega_P$, we have $T \notin \Omega_P \Rightarrow$ by induction hypothesis, $v_P(A) > a_{l-1} \Rightarrow v_P(A) \geq a_l$. So $\widehat{P}$ is a monotonic partial extension of P, hence of P_0. Now (2) holds for $\widehat{P}$, which is clear from the construction of $\Omega_{\widehat{P}}$. To get (3), consider first any $T \in \Omega_{\widehat{P}}$. It suffices to take $T \in \Omega_{\widehat{P}} \backslash \Omega_P$. Consider any $\Delta \in \mathbb{R}^{2^n}$ with $\Delta(T) \neq 0$. It suffices to take $\Delta(T) > 0$. By (X1), $\exists S \supseteq T$, $S \in \Omega_P$, such that $v_P(S) = v_P(T)$; so either $\Delta(S) = 0$ which directly implies $v_P + \Delta$ not monotonic, or $\Delta(S) \neq 0 \Rightarrow$ one of $v_P \pm \Delta$ is not a monotonic partial extension of P_0, since $S \in \Omega_P$.

Similar arguments show that if $P \in \mathcal{P}_l$ satisfies the claim then so does any $\bar{P} \in \mathcal{P}_l$ arising from P via (X2). For (1), consider any $T \in 2^{T^*} \backslash \Omega_P$ where $\max\{v_P(S) \mid S \subset T^*, S \in \Omega_P\} = a_l$; and apply the reasoning of the previous paragraph, with T^* and $\bar{P}$ in place of S and $\widehat{P}$. As before, (2) is clear. For (3), it suffices to consider any $T \in \Omega_{\bar{P}} \backslash \Omega_P$, any $\Delta \in \mathbb{R}^{2^n}$ with $\Delta(T) > 0$. Either $T = T^*$ for some T^* as defined in the statement of the theorem, in which case $\exists S \subseteq T$, $S \in \Omega_P$, such that $w(S) = w(T)$, so $w - \Delta \notin \mathfrak{M}(P_0)$; or $T \subset T^*$ for some T^* as defined in the statement of the theorem, in which case $w(T^*) = w(T)$, so $w + \Delta \notin \mathfrak{M}(P_0)$. This completes the induction. □

Claim 2: $\mathcal{P}_k \supseteq ex(\mathfrak{M}(P_0))$.

Proof: Some notation: For v a game or PDG, $a \in \mathbb{R}$, let $v^{-1}(a) = \{S \mid v(S) \text{ defined and } = a\}$.

Pick any $(N, w) \in ex(\mathfrak{M}(P_0))$. We show that for $i = 0, 1, 2, \cdots, k$, $\exists P \in \mathcal{P}_i$ such that $\forall j \leq i$, $w^{-1}(a_j) = v_P^{-1}(a_j)$. The claim would follow from the case $i = k$. Induct on i. The proposition holds for $i = 0$; assume it does for all $i < l$. By induction hypothesis $\exists Q \in \mathcal{P}_{l-1}$ such that $\forall j \leq l-1$, $w^{-1}(a_j) = v_Q^{-1}(a_j)$. We'll be done if we can construct $P \in \mathcal{P}_l$ partially extending Q and satisfying $w^{-1}(a_l) = v_P^{-1}(a_l)$.

An alternative characterization of $w^{-1}(a_l)$ is useful. We claim $w^{-1}(a_l) = \mathfrak{F}$, where $\mathfrak{F} \subseteq 2^N$ is defined by $\mathfrak{F} = \bigcup_{i \geq 0} \mathfrak{F}_i$ where

$$\mathfrak{F}_0 = w^{-1}(a_l) \cap \Omega_{P_0}$$

$$\mathfrak{F}_{i+1} = \mathfrak{F}_i \cup \underline{\mathfrak{F}}_i \cup \overline{\mathfrak{F}}_i \qquad \text{where} \qquad \underline{\mathfrak{F}}_i = \left\{ T \in w^{-1}(a_l) \mid \exists S \in \mathfrak{F}_i\text{: } T \subseteq S \right\}$$

$$\overline{\mathfrak{F}}_i = \left\{ T \in w^{-1}(a_l) \mid \exists S \in \mathfrak{F}_i\text{: } T \supseteq S \right\}$$

Suppose $w^{-1}(a_l) \neq \mathfrak{F}$. Let $\Delta(S) = \begin{cases} \varepsilon \text{ if } S \in w^{-1}(a_l) \setminus \mathfrak{F} \\ 0 \text{ if } S \notin w^{-1}(a_l) \setminus \mathfrak{F} \end{cases}$ where $\varepsilon = \min\{ |w(A) - w(B)| \mid w(A) \neq w(B) \}$.

Then $w \pm \Delta$ are distinct elements of $\mathfrak{M}(P_0)$ which sum to $2w$, contradicting $(N, w) \in ex(\mathfrak{M}(P_0))$.

Also, we may write $\mathfrak{F} = \bigcup_{i=0}^{m} \mathfrak{F}_i$ for some finite m.

Apply (X1) to Q to get a new PDG $Q_0 \in \mathfrak{P}_l$. Construct PDGs $Q_1, Q_2, \cdots Q_m$ as follows. Given Q_i define Q_{i+1} by successive applications of (X2) to Q_i, where we let T^* in the statement of (X2) range successively over all $T \in \overline{\mathfrak{F}}_i$ whose worths in Q_{i+1} are yet undetermined. We show that Q_m is the desired P partially extending Q.

Clearly $\mathfrak{F}_0 \subseteq v_{Q_0}^{-1}(a_l) \subseteq \mathfrak{F}$. Assuming $\mathfrak{F}_i \subseteq v_{Q_i}^{-1}(a_l) \subseteq \mathfrak{F}$, it follows that $\mathfrak{F}_{i+1} \subseteq v_{Q_{i+1}}^{-1}(a_l) \subseteq \mathfrak{F}$. The first inclusion is because for all $S \in \mathfrak{F}_i$ we have $2^S \setminus \Omega_Q \subseteq v_{Q_i}^{-1}(a_l) \subseteq v_{Q_{i+1}}^{-1}(a_l) \Rightarrow \underline{\mathfrak{F}}_i \subseteq v_{Q_{i+1}}^{-1}(a_l)$; the second inclusion is because any $S \in v_{Q_{i+1}}^{-1}(a_l)$ satisfies $S \subseteq T \in \overline{\mathfrak{F}}_i$ for some T, so $S \in \mathfrak{F}_{i+2}$. By induction, $\mathfrak{F}_i \subseteq v_{Q_i}^{-1}(a_l) \subseteq \mathfrak{F}$ for all i. So $\bigcup_{i \geq 0} \mathfrak{F}_i \subseteq \bigcup_{i \geq 0} v_{Q_i}^{-1}(a_l) \subseteq \mathfrak{F} \Rightarrow \mathfrak{F} = v_{Q_m}^{-1}(a_l)$. This completes the proof of Claim 2. $\square$

Thus $\mathfrak{P}_k = ex(\mathfrak{M}(P_0))$. $\square$

4 References

Letscher, D. (1990), "The Shapley Value on Partially Defined Games," manuscript.

Ventrudo, T., and J. Wallman (1991), "Finding a Value on Partially Defined Games," manuscript.