Axioms And Values On Partition Function Form Games

Patrice McCaulley

Final Report

REU Cooperative Game Theory

July 20, 1990

Introduction

Games in partition function form were first introduced in 1963 by Lucas and Thrall as a way of generalizing games in characteristic function form. In 1977, Myerson uncovered a value for games in partition function form which satisfies the extensions of the basic axioms which characterize the Shapley value in characteristic function form. Bolger derived a class of efficient values for partition function form games in 1987. Lastly, in 1989 Bolger arrived at a unique, recursively defined value for partition function form games which is characterized differently ~~than~~ from Myerson's value.

The outline of this paper is as follows:

1) a review of the basics of partition function form notation and meaning.
2) a description of thirteen axioms and six definitions in partition function form.
3) a derivation of N = 1,2,3,4 player value(s) using the axioms of efficiency, additivity, symmetry, and dummy.
4) an explanation of what, if any, effect other axioms have when applied to the values derived in part three, including how the values derived by Myerson(1977) and Bolger(1989) relate to the values derived in this paper.

Games in Partition Function Form

$N = \{1,2,\ldots,n\}$ is the set of players in a n person game.

$Cl = \{S \mid S \subseteq N, S \neq \emptyset\}$ is the set of coalitions of N.

PT = set of partitions of N:

$\{S^1,\ldots,S^m\} \in PT$ iff

$S^1 \cup \ldots \cup S^m = N$, $\forall j\ S^j \neq \emptyset$, $\forall k\ S^k \cap S^j = \emptyset$ if $k \neq j$.

ECL = set of embedded coalitions: $\{(S,P) \mid S \in P \in PT\}$.

A game in partition function form is any $W \in \mathbb{R}^{ECL}$.

$W(S;P)$ is the amount S would receive if partition P formed.

$\phi(W)$ is a payoff vector for the game W.

$\phi_i(W)$ is the payoff or allocation to player i on game W.

Definitions and Axioms in Partition Function Form

Each axiom that follows has a descriptive title followed by what it means for a value function ϕ to satisfy that axiom. For notational purposes, S and T are subsets of N, P and Q are partitions of N, and W and V are games on N.

Definition of a permutation π:

more should be defined here. — follow Myerson, perhaps.

$(\pi \circ W)[\pi(S;P)] = W(S;P) \quad \forall (S;P) \in ECL$

Axiom 1) **Symmetry:** means that the payoff to a player will not change if the names of the players are permuted. Hence $\forall j \in N$ and for every game W, $\phi_j(W) = \phi_{\pi(j)}(\pi \circ W)$.

Axiom 2) **Additivity:** $\forall W^1, W^2 \in R^{ECL}$, $\phi(W^1+W^2) = \phi(W^1) + \phi(W^2)$.

Axiom 3) **Efficiency:** $\phi_1(W) + \ldots + \phi_n(W) = W(N;N)$.

Def. 1: player j is a dummy in game W if $W(S;P) = W(S-\{j\};Q)$ for each nontrivial $(S;P) \in ECL$, where Q is a partition resulting from the removal of j to another, possibly empty, set of P.

Axiom 4) **Dummy:** If player j is a dummy in game W, then $\phi_j(W) = 0$.

Definitions 2,3 and axiom 5 are from Myerson(1977).

Def. 2: For any $P \in PT$ and $Q \in PT$, $P \wedge Q \in PT$ is defined as:

$P \wedge Q = \{S \cap T \mid S \in P,\ T \in Q,\ S \cap T \neq \emptyset\}$.

Def. 3: Given $W \in R^{ECL}$ and $S \in CL$, S is a carrier of W iff

$W(T;Q) = W(S \cap T; Q \wedge \{S, N\backslash S\})$, $\forall (T;Q) \in ECL$.

Axiom 5) **Carrier:** $\forall W \in R^{ECL}$, $\forall S \in ECL$, if S is a carrier of W, then $\Sigma_{n \in S}\, \phi_n(W) = W(N;N)$.

Definition 4 and axiom 6 are from Bolger(1987).

Def. 4: For a fixed $(S;P)$ let $W = W^{(S;P)}$ and for $d \notin N$, define the dummy extension W^d on $N \cup \{d\}$ by:

(i) $W^d(S;P^d) = 1$ for each partition P^d of $N \cup \{d\}$ obtained by placing d in some set of P other than S, or in a set by itself.

(ii) $W^d(S \cup \{d\};(P-\{S\}) \cup (S \cup \{d\})) = 1$.

(iii) $W^d(T;Q) = 0$ for all other $(T;Q) \in ECL$ where Q is a partition of $N \cup \{d\}$.

Axiom 6) **Dummy Independent:** For every game W on N, $\forall\, d \notin N$, and $\forall\, i \in N$, $\phi_i(W^d) = \phi_i(W)$.

A more general definition is needed. Try: Let w be a game on N. The dummy extension of W is the game W^d defined by $W^d(S;P) = W(S-\{d\}; \{R-\{d\} : R \in P\})$ for all ECLs

Axiom 7) **Individual Rationality:** For every game W,

$\phi_i(W) \geq$ ~~$W(i;\{i, N-i\})$~~ $W(\{i\};\{\{i\}, N-\{i\}\})$, $\forall\, i \in N$.

Don't "abuse notation" in this section.

Axiom 8) **Aggregate Monotonicity:** If $W(N;N) \geq V(N;N)$ and $W(S;P) = V(S;P)$, $\forall\ S \subsetneq N$ and $P \neq N$, then $\phi_i(W) \geq \phi_i(V)$ for all $i \in N$.

Axiom 9) **Group Monotonicity:** If $W(S;P) \geq V(S;P)$ and $W(T;Q) = V(T;Q)$, $\forall\ (T;Q) \neq (S;P)$, then $\phi_i(W) \geq \phi_i(V)$, $\forall\ i \in S$.

Axiom 10) **Complementary Group Monotonicity:** If $W(S;P) \geq V(S;P)$ and $W(T;Q) = V(T;Q)$, $\forall\ (T;Q) \neq (S;P)$, then $\phi_i(W) \leq \phi_i(V)$, $\forall\ i \notin S$.

Def. 5: In a game W, the marginals for player i are the quantities $W(S;P) - W(S-\{i\};Q)$, where $i \in S$ and Q is a partition resulting from starting with partition P and moving i from S to another, possibly empty set in P.

Axiom 11) **Strong Monotonicity:** If each marginal for player i on game W is greater than or equal to the corresponding marginal for player i on game V, then $\phi_i(W) \geq \phi_i(V)$.

Axiom 12) **Marginalist:** If the corresponding marginals for player i are the same on two games W and V, then the allocation ϕ_i should be the same for both games, $\phi_i(W) = \phi_i(V)$.

Axiom 13) **Bolger(1989) axiom:** Let $i \in N$. If for each partition Q, where $T \in Q$, $i \in T$ and ~~P is a~~ the summations are over all partitions P of N that can be obtained from Q by moving i from T into another, possibly empty, set we have

$$\sum_{\cancel{s \in Q, s \neq T}\ Q} [W(T;Q) - W(T-\{i\},P)] = \sum_{\cancel{s \in Q, s \neq T}\ Q} [V(T;Q) - V(T-\{i\},P)],$$

then $\phi_i(W) = \phi_i(V)$.

For brevity, ECL's will be written with set brackets and commas inside coalitions removed. For example, $W(\{1,2\};\{\{1,2\},\{3\},\{4\}\})$ will be written $W(12;12,3,4)$.

Results

In partition function form there is a unique value, ϕ, for $N = 1,2$ which satisfies the axioms which uniquely determine the Shapley value on characteristic function form games, which are symmetry, additivity, efficiency, and dummy (axioms 1,2,3,4 listed above). However, for $N > 2$, a class of values will satisfy these four axioms.

For $N = 1$, $\phi_1 = W(1;1)$.

For $N = 2$, $\phi_1 = (1/2)\,W(1;1,2) + (1/2)[W(12;12) - W(2;1,2)]$.

The $N = 3$ case is discussed in detail next, by forming a basis for all three player partition function form games using unanimity games and applying axioms 1,2,3,4 from above.

rearranging these rows & reversing $W^5 W^6 W^7$ with $W^8 W^9 W^{10}$ would clarify the upper triangularity of the matrix.

Three Person game

S;Q	W^1	W^2	W^3	W^4	W^5	W^6	W^7	W^8	W^9	W^{10}
{123};{123}	1	1	1	1	1	1	1	1	1	1
{1};{1,23}	0	0	0	0	1	0	0	0	0	0
{23};{1,23}	0	1	0	0	0	1	1	0	1	1
{2};{2,13}	0	0	0	0	0	1	0	0	0	0
{13};{2,13}	0	0	1	0	1	0	1	1	0	1
{3};{3,12}	0	0	0	0	0	0	1	0	0	0
{12};{3,12}	0	0	0	1	1	1	0	1	1	0
{1};{1,2,3}	0	0	0	0	1	0	0	1	0	0
{2};{1,2,3}	0	0	0	0	0	1	0	0	1	0
{3};{1,2,3}	0	0	0	0	0	0	1	0	0	1
Player 1:	1/3	0	1/2	1/2	1	0	0	1−2Z	Z	Z
Player 2:	1/3	1/2	0	1/2	0	1	0	Z	1−2Z	Z
Player 3:	1/3	1/2	1/2	0	0	0	1	Z	Z	1−2Z

The unanimity games described above form a basis for all three person partition function form games because if we write a table with the ECLs in the same order across the top and down the side, and then fill in the unanimity games it looks like:

```
.... Embedded Coalitions ....
.
.  1 1 1 1 1 1 ...
E  0 1 1 1 1 1 ...
C  0 0 1 1 1 1 ...
L  0 0 0 1 1 1 ...
s  0 0 0 0 1 1 ...
.  . . . . . . ...
.  . . . . . . ...
```

It is clearly in upper triangular form so the unanimity games are linearly independent, and since the number of unanimity games is the same as the dimension, these unanimity games form a basis for all three person games.

The following is an explanation of the allocations listed above for each of the unanimity games:

Game W^1: All three players look alike so by applying the symmetry axiom each player should receive the same amount and then applying the efficiency axiom each player must receive 1/3.

Game W^2: Players 2 and 3 look alike so by the symmetry axiom they should receive the same amount. Player 1 is a dummy so by the dummy axiom he/she should receive nothing. Finally by utilizing the efficiency axiom, players 2 and three each get 1/2.

Games W^3 and W^4: These games are the same as W^2 except that the players have been permuted. In game W^2 the dummy player is 1, but in W^3 the dummy player is 2 so by symmetry in W^3 player 2 should receive nothing and players 1 and 3 should each get 1/2. The argument for W^4 is similar.

Game W^5: Players 2 and 3 are dummies so by the dummy axiom they receive nothing and player 1 gets 1 by efficiency.

Games W^6 and W^7: These games are permutations of game W^5. In game W^6 player 2 is the nondummy and gets 1. Similarly, in game W^7 player 3 is the nondummy and receives 1.

Game W^8: Players 2 and 3 look alike so by the symmetry axiom they should receive the same amount Z. By efficiency, player 1 gets 1-2Z. Using only axioms 1,2,3,4 there is no way to come up with a unique value for Z so it is left as a parameter.

Games W^9 and W^{10}: These are permutations of game W^8 and therefore the payoffs to the players should be permuted accordingly.

Using $W^1,\ldots,W^{10}$, and ten corresponding constants, $k_1,\ldots,k_{10}$ we can find a class of values for any three player game, W. Hence, the equations we need to solve are:

$$W(123;123) = k_1 + \ldots + k_{10}$$

$$W(1;1,23) = k_5$$

$$W(23;1,23) = k_2 + k_6 + k_7 + k_9 + k_{10}$$

$$W(2;2,13) = k_8$$

$$W(13;2,13) = k_3 + k_5 + k_7 + k_8 + k_{10}$$

$$W(3;3,12) = k_7$$

$$W(12;3,12) = k_4 + k_5 + k_6 + k_8 + k_9$$

$$W(1;1,2,3) = k_5 + k_8$$

$$W(2;1,2,3) = k_6 + k_9$$

$$W(3;1,2,3) = k_7 + k_{10}$$

After solving the above equations for $k_1,...,k_{10}$ we have a class of values with one parameter, Z, which satisfy the efficiency, additivity, symmetry, and dummy axioms.

$$\phi_1 = (1/3)k_1 + (1/2)k_3 + (1/2)k_4 + k_5 + (1-2Z)k_8$$
$$+ (Z)k_9 + (Z)k_{10}$$

After substitution of the values for $k_1,...,k_{10}$ into the above equation, the allocation to player one is as follows:

$$\phi_1 = (1/3)[W(123;123) - W(23;1,23)]$$
$$+(1/6 - Z)[W(13;13,2) - W(3;1,2,3)]$$
$$+(Z)[W(13;13,2) - (Z)W(3;12,3)]$$
$$+(1/6 - Z)[W(12;12,3) - W(2;1,2,3)]$$
$$+(Z)[W(12;12,3) - W(2;13,2)]$$
$$+(1/3 - 2Z)W(1;1,2,3) + (2Z)W(1;1,23)$$

The application of some of the other axioms given above results in either a restriction on the possible values of Z, or in a unique value for Z. If we look at the same four axioms that characterized the value above, which are efficiency, additivity, symmetry, and dummy, and then add on axiom 9,10, or 11, group monotonicity, complementary group monotonicity, or strong monotonicity, the restriction that $0 \le Z \le 1/6$ results. Aggregate monotonicity, axiom 8, places no restrictions on Z, because the coefficient of W(123;123) is a non-negative constant, so no matter what Z is chosen, the allocation satisfies axiom 8. The dummy independent axiom, axiom 6, added to axioms 1,2,3,4 does not put any restrictions on Z in the three player game. Since $\phi_i(W)$ is just the sum of the marginals of i in game W, it satisfies marginalist, axiom 12, no matter what Z is. Axioms 1,2,3,4 and axiom 13, Bolger(1989), give a unique value with Z = 1/12. This is the three player value which was derived by Bolger in his 1989 paper, and it clearly satisfies group, complementary, and strong monotonicity. True, but why?

The carrier axiom is equivalent to the efficiency and dummy axioms on characteristic function form games, but on partition function form games it places more restrictions on an allocation. Consequently, if the carrier axiom is added to axioms 1,2,3,4 it results in a unique value for Z, which is Z = 1/3. This three player value is the one derived by Myerson in his 1977 paper, where he characterizes a value with only three axioms: additivity, symmetry, and carrier. It is clear from above that this allocation with Z = 1/3 is not group monotonic, not complementary group monotonic, and not stongly monotonic. why?

Next, is the derivation of the N = 4 formula. I will list the seven different **types** of four player unanimity games and the corresponding allocations using axioms 1,2,3,4. The thirty games not listed can be found by switching the names of the players. Any W(S;P) not specifically listed has value zero.

1) W(1234;1234) = 1.

$\phi_i(W) = 1/4, \ \forall \ i \in N.$

2) W(1234;1234) = W(123;123,4) = 1.

$\phi_i(W) = 1/3, \ \forall \ i \in N-\{4\}, \ \phi_4(W) = 0.$

3) W(1234;1234) = W(123;123,4) = W(124;124,3) = W(12;12,3,4) = 1.

$\phi_1(W) = \phi_2(W) = 1/2 - A$, and $\phi_3(W) = \phi_4(W) = A.$

4) $W(1234;1234) = W(123;123,4) = W(124;124,3) = 1$, and

$W(12;12,3,4) = W(12;12,34) = 1$.

$\phi_1(W) = \phi_2(W) = 1/2$, and $\phi_3(W) = \phi_4(W) = 0$.

5) $W(1234;1234) = W(123;123,4) = W(124;124,3) = W(134;134,2) = 1$,

$W(12;12,3,4) = W(13;13,2,4) = W(14;14,2,3) = 1$, and

$W(1;1,2,3,4) = 1$.

$\phi_1(W) = 1 - 3B$, and $\phi_i(W) = B$, $\forall\ i \in N-\{1\}$.

6) $W(1234;1234) = W(123;123,4) = W(124;124,3) = W(134;134,2) = 1$,

$W(12;12,3,4) = W(13;13,2,4) = W(14;14,2,3) = 1$,

$W(12;12,34) = \cancel{W(13;13,24) = W(14;14,23)} = 1$, and

$W(1;1,2,3,4) = W(1;1,2,34) = 1$.

$\phi_1(W) = 1 - C - 2D$, $\phi_2(W) = C$, and $\phi_3(W) = \phi_4(W) = D$.

7) $W(1234;1234) = W(123;123,4) = W(124;124,3) = W(134;134,2) = 1$,

$W(12;12,3,4) = W(13;13,2,4) = W(14;14,2,3) = 1$,

$W(12;12,34) = W(13;13,24) = W(14;14,23) = 1$,

$W(1;1,2,3,4) = W(1;1,2,34) = W(1;1,3,24) = 1$, and

$W(1;1,4,23) = W(1;1,234) = 1$.

$\phi_1(W) = 1$ and $\phi_i(W) = 0$, $\forall\ i \in N-\{1\}$.

Using these types of unanimity games and the same procedure as described above for the three player case, we get the following allocation for player one in a four player game:

$\phi_1(W) = [3C+6(A-B+D)]\ W(1;1,234) + \cancel{[-3A+3B-C-2D]\ W(1;1,2,34)}$

$W(1;1,2,34)+$

$+ [-3A+3B-C-2D]\ \{W(1;3,24)+W(1;1,4,23)\}$

new line $\qquad +W(14;14,23)$

$+ [1/4-3B+3A]\ W(1;1,2,3,4) + [A]\ \{W(12;12,34)+W(13;13,24)\}$

$\cancel{+ [A]\ W(14;14,23)} + [A-B+C]\ \{W(2;2,1,34)+W(3;3,1,24)\}$

$\cancel{+ [A-B+C]\ W(4;4,1,23)} + \cancel{[(1/12)\ A]\ W(12;12,3,4)} \qquad +W(4;4,1,23)$

$W(12;12,3,4)+$

$$+ [(1/12)-A]\ \{W(13;13,2,4)+W(14;14,2,3)\}$$

$$+ [B-A-(1/12)]\ \{W(2;2,1,3,4)+W(3;3,1,2,4)+W(4;4,1,2,3)\}$$

$$+ (1/12)\ \{W(123;123,4)+W(124;124,3)+W(134;134,2)\}$$

$$+ [A-(1/12)]\ \{W(23;23,1,4)+W(24;24,1,3)+W(34;34,1,2)\}$$

$$+ [1/4]\ \{W(1234;1234)-W(234;234,1)\}$$

$$+ [-2A+2B-C-2D]\ \{W(2;2,134)+W(3;3,124)+W(4;4,123)\}$$

$$+ [A-B+D]\ \{W(2;2,3,14)+W(2;2,4,13)+W(3;3,2,14)\}$$

$$+ [A-B+D]\ \{W(3;3,4,12)+W(4;4,2,13)+W(4;4,3,12)\}$$

$$+ [-A]\ \{W(23;23,14)+W(34;34,12)+W(24;24,13)\}$$

Be more specific.

This allocation has four parameters and it can be verified easily by the reader that it satisfies the efficiency, symmetry, and additivity axioms, but it does not satisfy the dummy axiom. Applying the dummy axiom to the allocation results in a relation between two of the parameters, for this allocation method to satisfy the dummy property. The relation is $D = B/2$, and therefore the allocation now only has three parameters and it satisfies axioms 1,2,3,4. Since ϕ satisfies additivity and dummy, it also satisfies the marginalist axiom and therefore ϕ can be written as a sum of its marginals as follows:

$$\phi_1(W) = [3C+6(A-B/2)]\ W(1;1,234)$$

$$+ [-3A+2B-C]\ W(1;1,2,34)$$

$$+ [-3A+2B-C]\ W(1;1,3,24)$$

$$+ [-3A+2B-C]\ W(1;1,4,23)$$

$$+ [1/4-3B+3A]\ W(1;1,2,3,4)$$

$$+ [B-A-C]\ [W(12;12,34) - W(2;2,1,34)]$$

$$+ [2A-B+C]\ [W(12;12,34) - W(2;2,134)]$$

$$+ [A-B+(1/12)]\ [W(12;12,3,4) - W(2;2,1,3,4)]$$

$$+ [(B/2)-A]\ [W(12;12,3,4) - W(2;2,3,14)]$$

$$+ [(B/2)-A]\ [W(12;12,3,4) - W(2;2,4,13)]$$

$$+ [B-A-C]\ [W(13;13,24) - W(3;3,1,24)]$$

$$+ [2A-B+C]\ [W(13;13,24) - W(3;3,124)]$$

$$+ [A-B+(1/12)]\ [W(13;13,2,4) - W(3;3,1,2,4)]$$

$$+ [(B/2)-A]\ [W(13;13,2,4) - W(3;3,2,14)]$$

$$+ [(B/2)-A]\ [W(13;13,2,4) - W(3;3,4,12)]$$

$$+ [B-A+C]\ [W(14;14,23) - W(4;4,1,23)]$$

$$+ [2A-B+C]\ [W(14;14,23) - W(4;4,123)]$$

$$+ [A-B+(1/12)]\ [W(14;14,2,3) - W(4;4,1,2,3)]$$

$$+ [(B/2)-A]\ [W(14;14,2,3) - W(4;4,2,13)]$$

$$+ [(B/2)-A]\ [W(14;14,2,3) - W(4;4,3,12)]$$

$$+ [(1/12)-A]\ [W(123;123,4) - W(23;23,1,4)]$$

$$+ [A]\ [W(123;123,4) - W(23;23,14)]$$

$$+ [(1/12)-A]\ [W(124;124,3) - W(24;24,1,3)]$$

$$+ [A]\ [W(124;124,3) - W(24;24,13)]$$

$$+ [(1/12)-A]\ [W(134;134,2) - W(34;34,1,2)]$$

$$+ [A]\ [W(134;134,2) - W(34;34,12)]$$

$$+ [1/4]\ [W(1234;1234) - W(234;234,1)]$$

Lastly, the effects of some axioms on the three parameters of this allocation method for the four player partition function form game will be examined. For this value to be group, complementary, or strongly monotonic, axioms 9, 10, 11, respectively, there will be the following three restrictions on the parameters: $0 \le A \le 1/12$, $B \le (1/12) + A$, and $C \ge B - 2A$.

not quite

Marginalist, axiom 12, is clearly satisfied no matter what A,B,C are, because this allocation is only a sum of the marginals. Aggregate monotonicity, axiom 8, is also satisfied no matter what A,B,C are because the coefficient of W(1234;1234) is a non-negative constant. It can be easily verified by the reader that dummy independent, axiom 6, is satisfied by this three parameter class of values for the four player game. The carrier axiom added to axioms 1,2,3,4 just as in the three player allocation, results in a unique value for the four player game, A = 1/4, B = 1/6, and C = -1/3, which is the four player value Myerson derived in his 1977 paper. From the restrictions given above for group, complementary, and strong monotonicity, it is clear that Myerson's four player value does not satisfy these axioms.

I don't believe this. Dummy independence should force a relationship among A,B & C.

How did you obtain this?

Myerson's value is not individually rational!

References

Bolger,E.M.: Values for Games in Partition Function Form. Society for Industrial and Applied Mathematics,1987,8:460-466.

Bolger,E.M.: A Set of Axioms for A Value for Partition Function Games.International Journal of Game Theory, 1989,18:37-44.

Lucas,W.F.,and R.M. Thrall: n-Person Games in Partition Function Form.Naval Research Logistics Quarterly X,1963,281-298.

Myerson,R.B.: Values of Games in Partition Function Form. International Journal of Game Theory,1977,6:23-31.

DREW UNIVERSITY
Department of Mathematics
and Computer Science
College of Liberal Arts
Madison, New Jersey 07940-4037
(201) 408-3161

August 20, 1990

Patrice McCaulley
9 South 30th Ave.
Longport, NJ 08403

Dear Patrice,

I hope that you have been able to relax after eight weeks of mathematics. Unfortunately, I have had administrative work for the Council on Undergraduate Research, lectures for the New Jersey Governor's School, and software library development for Introductory Statistics. Jeanne and I took a few hours off for our anniversary, but now I'm back finishing up REU stuff.

I have enclosed (1) a request for an evaluation of the program, (2) the original of your REU report upon which I have written a number of suggestions, and (3) a copy of participants' whereabouts. Please return your evaluation to me by September 17. If you would like a revised copy of your report to be sent to the NSF and other interested persons, please return your revision by September 17. I have held off sending out copies of student reports pending each student's decision whether or not to revise. There is no requirement to revise your report; it is up to you based upon your time and interest. In late September, I will send you copies of the other reports.

There is quite a lot in your report (it took me a full day, and now into the wee hours to digest it all). You have made a good start in understanding what is happening on partition function form games. Before I tell you about the problems, I want to look at the motivation for this work. The Shapley value on TU games satisfy all of the axioms you list (individual rationality holds on superadditive games) and is characterized by efficiency, symmetry, additivity and dummy. Nonetheless, the axioms listed are inconsistent on partition function form games (for example, Myerson's value is not group monotone), and the four Shapley axioms do not characterize a unique value on partition function form games. This difference between TU games and the more general partition function form games is the first motivation for this work. Shapley's orginal characterization need only consider superadditive games; however, both Bolger and Myerson use games that are not superadditive as defined below. So, the second motivation is to see what can be done when only superadditive games are considered. I think that this would be particularly relevant to cost allocation problems.

Definitions. A game w is *weakly superadditive* if $w(S; P) + w(T; P) \leq w(S \cup T; P - \{S,T\} \cup \{S \cup T\})$ for all $S, T \in P \in PT$. A game w is *opposition monotonic* if $w(S; P) \leq w(S; Q)$ for all $S \in P \in PT$ and refinements $Q \in PT$. A game w is *superadditive* if w is both weakly superadditive and opposition monotonic. The *unanimity game* on $(S; P) \in ECL$, denoted by $w[S; P; c]$, is the minimum nonnegative superadditive game w satisfying $w(S; P) = c$.

Now I can describe the major problem in your report. The type 6 four-player game is *not* a unanimity game. The unanimity game on $(1; 1, 2, 34)$ has $w(13; 13, 24) = w(14; 14, 23) = 0$. I verified your formulas (note that the suggested changes in the first formula are stylistic, not substantive). The type 6 game you used is superadditive, so your characterization holds on superadditive games; however, we wanted to have a formula in terms of parameters based on unanimity games. Don't despair! I think that the formula based on unanimity games can be obtained fairly easily from the formula you already derived. We just need to calculate the values for the players in the unanimity game on $(1; 1, 2, 34)$ using your formula; set these equal to the new parameters; solve for C and D in terms of A, B, and the new parameters; and eliminate C and D in your formula.

We had some trouble applying the other axioms, and your report contains errors and insufficient explanations. First, group, complementary group, and strong monotonicity, respectively, hold if and only if the coefficients of $W(S; P)$ are nonnegative whenever S contains 1, the coefficients of $W(S; P)$ are nonpositive whenever S does not contain 1, and the coefficient of each marginal for 1 is nonnegative. For four player games, the three notions of monotonicity are not equivalent. For strong monotonicity, I obtained the restrictions $0 \leq A \leq 1/12$, $2A \leq B \leq A + 1/12$, and $B - 2A \leq C \leq B - A$. Second, in order to apply dummy independence, you need to assume that player 4 is a dummy and reduce the four player formula to a three player formula. This formula should match your three player formula after some restriction is placed on Z, A, B, and C. Third, axiom 13 (TU marginalist is a possible name) holds in the presence of the four Shapley axioms if and only if the coefficients for marginals based on the same ECL are equal. In the three player case, this means that $1/6 - Z = Z$ and $1/3 - 2Z = 2Z$.

Finally, I am really puzzled by the carrier axiom. In the three player case, the game $w(123; 123) = w(1; 1, 23) = 1$ has 1 as a carrier, and so $1 = \phi_1(w) = 1/3 + 2Z$ which implies $Z = 1/3$. In the four player case, the game $w(1234; 1234) = w(12; 12, 34) = 1$ has 12 as a carrier, and so $0 = \phi_3(w) = 1/4 - A$ and $1/2 = \phi_1(w) = 1/4 + B - A - C$ which imply that $A = 1/4$ and $B - C = 1/2$. These formulas agree with the values your report states. But $w(1234; 1234) =$

$w(1; 1, 234) = 1$ has 1 as a carrier, and so $1 = \phi_1(w) = 1/4 + 3C + 6A - 3B = 1/4$ by the earlier results. This seems to indicate that the value does not exist! What am I doing wrong? Or is Myerson's proof incorrect?

The next step is to "redo" the four player case as suggested above and apply all of the other axioms carefully. Perhaps we can get some more insight into the general case. Another task is to consider the general interrelationships of the various axioms. In particular, I conjecture that (1) carrier $\Rightarrow$ efficient and dummy, (2) dummy and additive $\Rightarrow$ marginalist, (3) dummy and additive $\Rightarrow$ [strong monotone $\Leftrightarrow$ group and complementary group monotone], and (4) efficient, symmetric and additive $\Rightarrow$ aggregate monotone. Finally, it would be interesting to see whether carrier is equivalent to efficient and dummy on the restricted class of superadditive games.

I will close with a few words about recommendations. I would be happy to write you a recommendation for graduate study or employment upon request. It is my policy to always share with you a copy of my letter of recommendation for you. If there is sufficient time between your request and the receipt deadline, I will send you a first draft for comment. You received two very strong letters of recommendation when you applied to the REU program. The letter from Marijke van Rossum was the most persuasive to me for the REU program; however, the letter from Carl McCarty was the more specific about your mathematical ability which is perhaps more important to a graduate school.

Good luck digesting all of this! Give me a call if I can be of help.

Sincerely,

David Housman