Cooperative Games in Partition Function Form

by

Luz E. Pinzon

1992 REU Final Report
Drew University
August 14, 1992

Partition Function Form (PFF) Games

Introduction

Games in partition function form differ from those in coalition function form in that the worth assigned to each coalition depends on how the rest of the players collude. Although little is known about the "reasonableness" of values for coalition function form games, natural extensions of the Shapley value and the nucleolus are considered for PFF games. In my research various concepts of reasonable outcomes are defined for these games based on those defined by Milnor for coalition function form games. Also an allocation method analogous to the nucleolus is sought for PFF games.

Background Information

We begin with a set of players $N=\{1,2,\ldots,n\}$ and consider all coalitions $S \subseteq N$. A game W in partition function form assigns a real value, or worth, to each subgroup S depending on the partition to which S may belong. To define W formally, we first introduce the following sets:

- The set of possible coalitions,

  $CL = \{S \subseteq N : S \neq \emptyset\}$

- The set of possible partitions,

  PT = {P: P is a partition of N}

  $\{S_1, S_2, \ldots, S_k\}$ is a partition of N iff:

  - $\emptyset \neq S_i \subseteq N$, for each $i = 1,\ldots,k$.
  - $\forall i \in N$, $\exists k$ such that $i \in S_k$.
  - $S_i \cap S_j = \emptyset \quad \forall i \neq j$.

- The set of embedded coalitions,

  $ECL = \{(S;P) : S \in P \in PT\}$

Thus, a partition function form game on N is any $W \in R^{ECL}$ where W(S;P) is the amount that S would receive if partition P formed.

We denote an allocation vector for the game W by **x**(W), or simply **x** whenever W is understood, and we say that **x**(S) is the payoff corresponding to $S \subseteq N$. In this paper, we consider only **x** in $\overline{E}$, the set of all efficient outcomes. Formally, we require that

$$\mathbf{x} \in \overline{E} = \left\{x: \sum_{i=1}^{n} x(i) = W(N;N)\right\}.$$

Two well-known allocation methods for coalition function form games (games which may be considered PFF games satisfying $W(S;P)=\overline{W}(S)$ for all $(S;P) \in ECL$) are the Shapley value and the nucleolus. We shall focus on the study of the nucleolus, which is defined to be the value that minimizes the vector of "complaints" in the lexicographic sense, and extend its meaning for PFF games.

First, we say that a vector **x** is smaller than a vector **y** in the lexicographic sense, and write $\mathbf{x} \leq_{lex} \mathbf{y}$, if any one of the following holds:

(a) **x** = **y**, or

(b) in the first component they differ $x_i < y_i$, or

(c) there is no such **i** when **x** has less components than **y**.

For example, the following vectors are arranged in descending lexicographic order:

[1, -3], [1, -3, -3], [-1, -3], [-1, -3], [-3, -3].

We define the vector of **coalitional excesses** with respect to **x** and P to be,

$$e(x,P)=(W(S;P)-x(S): S \in P)$$

where all its components are arranged in descending order. (If we consider W(S;P) as the potential value for S in a given partition P, we can also refer to e(x,P) as coalitional "complaints.")

Now, we consider the vector of **embedded complaints** with respect to P,

$$E(x)=[\ e(x,P):\ P\in PT\ ]$$

where all its components are arranged in descending *lexicographic* order, and which we wish to minimize so that the maximum complaint is as small as possible. So, define μ to be the allocation such that

$$E(\mu) = \textit{lexicographic}\ \min[E(x): x \in \overline{E}]$$

Example 1.

A three-player game with N={i,j,k}:

For simplicity, we make use of abusive notation and ignore brackets and commas when listing the members of a coalition S, and also ignore brackets when listing the coalitions of a partition P of N.

We begin with the following PFF game.

S	P	v(S;P)
ijk	ijk	48
ij	ij, k	24
ik	ik, j	18
jk	jk, i	6
i	i, j, k	12
i	i, jk	0
j	j, i, k	6
j	j, ik	0
k	k, i, j	9
k	k, ij	0

Consider the allocation x(v) = (19, 13, 16). We have the following coalitional complaints:

P	e(x,P)
ij, k	(-8, -16)
ik, j	(-13, -17)
jk, i	(-19, -23)
i, j, k	(-7, -7, -7)

So the vector of embedded complaints - in decreasing lexicographic order - is

$$E(x) = [\ (-7, -7, -7),\ (-8, -16),\ (-13, -17),\ (-19, -23)\ ].$$

Can we do better? That is, can we find another allocation which improves any of the above complaints? Suppose y~~(v)~~ is such an allocation. Since -7 is the largest complaint, then any complaint with respect to y must be at most -7. In particular we must require:

$$(*)\qquad \begin{aligned} 12 - y(i) \le -7 &\iff y(i) \ge 19 \\ 6 - y(j) \le -7 &\iff y(j) \ge 13 \\ 9 - y(k) \le -7 &\iff y(k) \ge 16 \end{aligned}$$

These are the complaints when P={i, j, k}. But since y must be efficient, we must have that equality holds in (*). Thus, $x = y = \mu$.

The μ- value is not an extension of the nucleolus, however. To show this, we need only consider the following example.

Example 2.

A three-player game with N={i,j,k}.

S	P	v(S;P)
ijk	ijk	16
ij	ij, k	12
ik	ik, j	8
jk	jk, i	4
i	i, j, k	0
i	i, jk	0
j	j, i, k	0
j	j, ik	0
k	k, i, j	0
k	k, ij	0

Notice that v(S;P)=v(S) so that this is a game in coalition function form whose nucleolus is ν(v)=(9, 5, 2). The vector of embedded complaints with respect to ν is:

$$E(\nu) = [\ \overset{ij\ \ k}{(-2, -2)},\ \overset{k\ \ j\ \ i}{(-2, -5, -9)},\ \overset{ik\ \ j}{(-3, -5)},\ \overset{jk\ \ i}{(-3, -9)}\ ]$$

Can we do better? Suppose x~~(v)~~ is better in the sense that $E(x) \le E(\nu)$. Since -2 is the largest complaint, we need:

$$\left.\begin{array}{r} 12 - x(i) - x(j) \le -2 \\ - x(k) \le -2 \end{array}\right\} \quad \Leftrightarrow \quad \begin{array}{r} x(i) + x(j) \ge 14 \\ x(k) \ge 2 \end{array} \qquad (*)$$

Efficiency forces (*) to hold as equalities. That is, $x(i) + x(j) = 14$ and $x(k) = 2$. Notice that regardless of the values that x(i) and x(j) may take on, the first coalitional complaint above will not change. We need every other complaint to be less than or equal to -2. That is we need the following:

$x(i) \ge 2$
$x(j) \ge 2$

$8 - x(i) - x(k) \le -2$	$4 - x(j) - x(k) \le -2$
$\Leftrightarrow x(i) + 2 \ge 10$	$\Leftrightarrow x(j) + 2 \ge 6$
$\Leftrightarrow x(i) \ge 8$	$\Leftrightarrow x(j) \ge 4$

Thus $8 \le x(i) \le 10$ and $4 \le x(j) \le 6$. We cannot have $x(j) = 4$ for that gives x = (10, 4, 2) and we would be worse off as indicated by

$$E(x) = [\ (-2, -2),\ (-2, -4, -10),\ (-2, -10),\ (-4, -4)\ ].$$

If $x(j) = 5$, we would have x = ν. Now suppose $x(j) = 5 + \varepsilon$ where $0 < \varepsilon \le 1$, so that x = $(9 - \varepsilon,\ 5 + \varepsilon\ ,\ 2)$. This gives,

$$E(x) = [\ (-2, -2),\ (-2, -5 - \varepsilon, -9 + \varepsilon),\ (-3 + \varepsilon, -5 - \varepsilon),\ (-3 - \varepsilon, -9 + \varepsilon)\ ]$$

and E(x) < E(ν). Notice that the best we can do now is let $\varepsilon = 1$ so that μ = (8, 6, 2), and we have $\mu \ne \nu$.

A computation method for the μ-value is not yet known, or how much it might differ from the nucleolus in case the PFF game is actually a coalition function form game. In example 2, the μ-value was not very different from ν in the sense that the corresponding payoffs to each player do not differ by more than one unit. However a more significant difference might occur for games with a large number of players. Also, it is not clear that the natural extension of the nucleolus, defined to be the allocation $\overline{\nu}$ so that

$$\hat{E}(\overline{\nu}) = \min[\hat{E}(x) : x \in \overline{E}\ ] = \min\ [\ \langle v(S) - x(S) : S \in P \in PT \rangle : x \in \overline{E}],$$

would yield more desirable results. We do know about the μ-value that it is not always "reasonable" and that it satisfies the **Equal Treatment Property (ETP).** We shall explain reasonableness later and show the following result:

Theorem I: In a PFF game v, if players i and j are symmetric, then $\mu(i) = \mu(j)$.

Definition: Two players i and j are symmetric in a game v if

$$v(S \cup \{i\}) = v(S \cup \{j\}) \quad \forall S \subseteq N \setminus \{i, j\}.$$

Theorem I says that the μ-value satisfies the equal treatment property for it assigns the same payoff to players that have the same effect on the worth function.

Proof I:

Let $e_{x,P}(S)$ denote the componet in the vector e(x,P) given by v(S;P) - x(S). This is the complaint of coalition S in the vector of coalitional complaints with respect to x and a given partition $P \ni S$.

Suppose that the μ-value does not satisfy ETP and let $\mu(i) = \mu(j) + 2\varepsilon \ \ (\varepsilon > 0)$.

Now let $y \in \overline{E}$ be another allocation such that $y(i) = y(j) = \frac{1}{2}\, \mu(ij) = \mu(j) + \varepsilon$.
and $y(k) = \mu(k)$ for all $k \in N \setminus \{i, j\}$

We show that $E(y) <_{lex} E(\mu)$.

First, consider the possible "types" of partitions of N that we have:

(i) P_b: one in which both i and j belong to the same set S of some partition.

(ii) P_S: the partition of "singletons" in which every player works separately.

(iii) $\{P_1, P_2\}$: partitions that come in "pairs," where P_1 is a partition for which i and j belong to two distinct sets respectively, and P_2 is obtained from P_1 by permuting players i and j. More formally,

$$P_1 = \{S \cup \{i\}, T \cup \{j\}\} \cup Q$$
$$P_2 = \{S \cup \{j\}, T \cup \{i\}\} \cup Q$$

where Q is some partition of $N \setminus (S \cup T \cup \{i, j\})$. We say that $\{P_1, P_2\}$ is a pair of symmetric partitions with respect to i and j.

Notice that for each $\{P_1, P_2\}$ of type (iii) we have that, since y(i)=y(j), then $e(y, P_1) = e(y, P_2)$.

Now suppose that the pair $\{P_1, P_2\}$ gives the largest vector of coalitional complaints $e(y, P_1)$ over all pairs of type (iii). Then E(y) may take the form:

(a) $E(y) = \langle \ldots, e(y, P_1),^- e(y, P_2), \ldots, e(y,P_s), \ldots \rangle$ or

(b) $E(y) = \langle \ldots, e(y, P_s), \ldots, e(y,P_1),^- e(y, P_2), \ldots \rangle$.

Notice that since $\mu(ij) = y(ij)$ then for any P_b of type (i) we get:

$$e(\mu,P_b) = e(y,P_b).$$

Thus, in our comparison of $E(\mu)$ and E(y), we may ignore all vectors of coalitional complaints of the form $e(\mu,P_b)$ and $e(y,P_b)$ for they will yield no difference.

Assume that E(y) is of the form in (a). We can suppose without loss of generality that:

- in $e(y, P_1)$ we have $e_{y,P_1}(S \cup i) \geq e_{y,P_1}(T \cup j)$ and that
- in $e(y, P_2)$ we have $e_{y,P_2}(S \cup j) \geq e_{y,P_2}(T \cup i)$.

Note that $\forall R \in Q$, $\quad e_{\mu,P_1}(R) = e_{y,P_1}(R) = e_{y,P_2}(R) = e_{\mu,P_2}(R)$.

Now use $y(i) = y(j) = \mu(j) + \varepsilon$ and $y(i) = y(j) = \mu(i) - \varepsilon$ to compute:

$$\left.\begin{aligned} e_{\mu,P_1}(S \cup i) &= e_{y,P_1}(S \cup i) - \varepsilon \\ e_{\mu,P_1}(T \cup j) &= e_{y,P_1}(T \cup j) + \varepsilon \end{aligned}\right\}$$

$$\left.\begin{aligned} e_{\mu,P_2}(S \cup j) &= e_{y,P_2}(S \cup j) + \varepsilon \\ e_{\mu,P_2}(T \cup i) &= e_{y,P_2}(T \cup i) - \varepsilon \end{aligned}\right\}$$

Then,

$$e(y,P_1) = \langle \ldots, e_{y,P_1}(S \cup i), \ldots, e_{y,P_1}(T \cup j), \ldots \rangle =$$
$$e(y,P_2) = \langle \ldots, e_{y,P_2}(S \cup j), \ldots, e_{y,P_2}(T \cup i), \ldots \rangle$$

and using the above equations, we get

$$e(\mu,P_1) = \langle \ldots, e_{\mu,P_1}(S\cup i), \ldots, e_{\mu,P_1}(T\cup j), \ldots \rangle = \langle \ldots, e_{y,P}(S\cup i) - \varepsilon, \ldots, e_{y,P_1}(T\cup j) + \varepsilon, \ldots \rangle$$

and

$$e(\mu,P_2) = \langle \ldots, e_{\mu,P_2}(S\cup j), \ldots, e_{\mu,P_2}(T\cup i), \ldots \rangle = \langle \ldots, e_{y,P_2}(S\cup j) + \varepsilon, \ldots, e_{y,P_2}(T\cup i) - \varepsilon, \ldots \rangle$$

Notice that $e(y,P_1) = e(y,P_2) <_{lex} e(\mu, P_2)$. **(*)**

So, given a fixed partition P_2 whose coalitional complaints are the largest (over all partitions of type (iii)), we get that we are worse off with respect to μ as compared with y.

We now compare $e(y,P_s)$ and $e(\mu,P_s)$.

Since $\mu(i) = y(i) + \varepsilon$ and $\mu(j) = y(j) - \varepsilon$ we have

$$e(y,P_s) = \langle \ldots, e_{y,P_s}(j), = e_{y,P_s}(i), \ldots \rangle$$

$$e(\mu,P_s) = \langle \ldots, e_{y,P_s}(j) + \varepsilon, \ldots, e_{y,P_s}(i) - \varepsilon, \ldots \rangle$$

Notice that $e_{\mu,P_s}(k) = e_{y,P_s}(k) \quad \forall k \in N \setminus \{i, j\}$, and also

$$e_{\mu,P_s}(i) < e_{y,P_s}(i) = e_{y,P_s}(j) < e_{\mu,P_s}(j) \qquad (**)$$

Again, we are worse off with respect to μ as compared with y. Thus, E(y) < E(μ) which contradicts the definition of μ.

Similarly, for E(y) of the form (b), **(*)** and **(**)** imply that E(y) < E(μ), and Theorem I is established.

Reasonableness and PFF Games

We want to consider only a limited set of outcomes for PFF games; in particular, we wish to define "reasonable" classes of allocations for these games in the sense described by Milnor [1952]. Although little is known about Milnor's classes of reasonable outcomes for coalition function form games, we formulate parallel definitions for PFF games and study them in detail.

- The set of all possible efficient outcomes:

$$\bar{E} = \left\{x \in R^n : \sum_{i \in N} x(i) = v(N;\varnothing)\right\}$$

- Class B of outcomes that do not exceed the maximum individual marginal contribution. First, we define the marginal contribution of player i to the embedded coalition (S;P) as follows:

$$M_i(S;P) = v(S;P) - v(S \setminus \{i\}; P[i,R])$$

where P[i,R] is the partition obtained from P by moving i from S to some other, possibly empty, set R of P. That is,

$$P[i,R] = P \setminus \{R, S\} \cup \{S \setminus \{i\},\ R \cup \{i\}\}, \qquad R \in P \setminus \{S\} \cup \{\varnothing\}.$$

Notice that if S does not contain i, then $M_i(S;P)=0$.

And so,

$$B = \left\{x \in \bar{E} : x(i) \le \max\left(M_i(S;P) : (S;P) \in ECL\right) : \forall i \in N\right\}.$$

- Class $\bar{D}$ of outcomes that assign reasonable amounts to every coalition S of N. We say that an amount δ **is reasonable** for coalition S if it is not unreasonable. On the other hand, we say that δ is **unreasonable** if N\S can somehow prevent S from getting δ. Formally, we require the following:

(1) $\exists P \in PT(S) = \{P \in PT : S \in P\}$ s.t. $v(S;P) < \delta$ and

(2) $\exists x \in \bar{E}$ s.t.

(i) $x(R) \le v(R;P) \quad \forall S \ne R \in P$

(ii) $x(T) > \max_{Q \in PT(S \cup T)} \left[v(S \cup T; Q) - \delta\right] \quad \forall \varnothing \ne T \subseteq N \setminus S$

Condition (1) above says that N\S must "partition" themselves so that S gets less than the δ it demands. Condition (2) says that this partition must be a "stable" one by requiring that N\S find an efficient allocation x so that: (i) the payoff to each member (coalition) of the partition in (1) be "feasible," and (ii) S be unable to disrupt the partition by causing defections to occur. This is possible if there is no incentive for any set of N\S to work together with S knowing S will keep δ.

And so, we define:

$$\overline{D} = \left\{x \in \overline{E}\colon x(S) \text{ is not unreasonable for any } S \subseteq N\right\}$$

We shall focus our attention to class $\overline{D}$.

Example 3.

Given any PFF game (N,v), we wish to know the reasonable payoffs that can be awarded to each coalition $S \subseteq N$. Reconsider the game in Example 1:

S	P	v(S;P)
ijk	ijk	48
ij	ij, k	24
ik	ik, j	18
jk	jk, i	6
i	i, j, k	12
i	i, jk	0
j	j, i, k	6
j	j, ik	0
k	k, i, j	9
k	k, ij	0

<u>Unreasonable δ for $S = \{i\}$</u>:

We assume $\delta > 0$ and take P={i, jk}.

Notation: we write v(S; P\{S}) instead of v(S; P).

We need $x \in \overline{E}$ such that

(i) $x(jk) \le v(jk;P) \quad \Leftrightarrow \quad x(jk) \le 6$

and

$$\text{(ii)}\quad \left.\begin{array}{l} x(j) > v(ij;k) - \delta \\ x(k) > v(ik;j) - \delta \\ x(jk) > v(ijk;\emptyset) - \delta \end{array}\right\} \Leftrightarrow \left\{\begin{array}{l} x(j) > 24 - \delta \\ x(k) > 18 - \delta \\ x(jk) > 48 - \delta \end{array}\right. \Leftrightarrow \left\{\begin{array}{l} x(jk) > 42 - 2\delta \\ x(jk) > 48 - \delta \end{array}\right.$$

In order to have such x, we need to have

$$\left.\begin{array}{l} 6 > 42 - 2\delta \\ 6 > 48 - \delta \end{array}\right\} \Leftrightarrow \left\{\begin{array}{l} \delta > 18 \\ \delta > 42 \end{array}\right.$$

And so, unreasonable payoffs for player i are any $\delta > 42$.

Similarly we can determine that unreasonable δ for player j are $\delta > 30$ and unreasonable δ for player k are $\delta > 24$.

<u>Unreasonable δ for $S = \{ij\}$</u>:

Assume $\delta > 24$ and take P={ij, k}.

We need $x \in \overline{E}$ such that:

(i) $x(k) \le v(k;ij) \quad \Leftrightarrow \quad x(k) \le 0$

(ii) $x(k) > v(ijk;\emptyset) - \delta \quad \Leftrightarrow \quad x(k) > 48 - \delta$

So we need $0 > 48 - \delta$, and any $\delta > 48$ is unreasonable for {ij}.

Similarly, we get that unreasonable δ for {ik} are $\delta > 48$, and unreasonable δ for {jk} are also $\delta > 48$.

Notice that the μ-value for this game, μ=(19, 13, 16), gives reasonable payoffs to each coalition $S \subseteq N$ so that the $\overline{D}$ set for (N,v) is nonempty.

However, the μ-value is not always reasonable. Consider the following game:

Example 4.

A three player game with N={i, j, k} and worth function given by:

S	P	v(S;P)
ijk	ijk	3
ij	ij, k	2
ik	ik, j	2
jk	jk, i	2
i	i, j, k	2
i	i, jk	0

S	P	v(S;P)
j	j, i, k	0
j	j, ik	0
k	k, i, j	0
k	k, ij	0

Here, μ=(1.75, 0.75, 0.75), but we get that unreasonable δ for any singleton are $\delta > 1$, and unreasonable δ for any pair are $\delta > 3$. So, for instance, μ(i), μ(ij), and μ(ik) are unreasonable payoffs; thus $\mu \notin \overline{D}$. Notice, however that $\overline{D}$ is nonempty for the allocation x=(1, 1, 1) is in it.

A natural question to ask is when, or for what games, is $\overline{D}$ is nonempty. In the case of coalition function form games, not much is known about the classes of reasonable outcomes defined by Milnor. In particular, it has not been determined if, or when, the analogous class D of reasonable outcomes is nonempty. In attempting to answer this question for class $\overline{D}$, we restrict ourselves to symmetric games in PFF.

Definition: A PFF game is symmetric if all the players are symmetric; ie, if permuting the players in any ECL does not change the worth function.

Consider the following game.

Example 5.

A symmetric game with N={i, j, k}.

S	P	v(S;P)
ijk	ijk	4
ij	ij, k	3
i	i, j, k	0
i	i, jk	2

We can compute the unreasonable values for each coalition and get that $\delta > 4$ is unreasonable for any one player {i}, and that $\delta > 2$ is unreasonable for any pair {ij}. Since all players are symmetric, if $\delta > 2$ is unreasonable for a pair {ij}, then $\delta > 1$ is unreasonable for each player {i}. But this implies that no efficient allocation will give reasonable payoffs to every coalition of N, and thus $\overline{D}$ is empty for this game.

Why did this happen? We take a closer look at the game above as well as the definition of $\overline{D}$. Recall that condition (2i) says that the partition P chosen in (1) is not to be disrupted by S. In our game, when S={i}, we have P={i, j, k}. But notice that, instead of i persuading either j or k to work with him, and causing "defections" to occur, it is more profitable for i to convince j and k to work together. This is <u>not</u> the kind of "disruption" of P that is considered in (2i). Also, notice that there is no incentive for certain coalitions to form, for some players can do better on their own. For instance, the grand coalition N is unlikely to form since the partition P={i, jk} is more profitable. Consider another example.

Example 6.

A symmetric game with N={i, j, k}.

S	P	v(S;P)
ijk	ijk	9
ij	ij, k	5
i	i, j, k	0
i	i, jk	4

Here we get that $\delta > 9$ is unreasonable for any singleton {i}, and that $\delta > 5$ is unreasonable for any pair {ij}. Since the game is symmetric, having $\delta > 5$ unreasonable for a pair {ij} implies that $\delta > 5/2$ is unreasonable for any singleton. Thus, again $\overline{D}$ is empty. Examining this game as we did before, we notice that, although there is never a disincentive for coalitions to form, condition (2i) in the definition of $\overline{D}$ does not "seem" to apply.

We now formalize the properties we saw in the two previous games.

Definition: A PFF game v is **superadditive** if, given any fixed partition P containing distinct sets S and S', we have

$$v(S;P) + v(S';P) \leq v(S \cup S';Q)$$

where Q is the partition obtained from P by joining S and S'; i.e.

$$Q = P \setminus \{S, S'\} \cup \{S \cup S'\}.$$

We say that in a superadditive game there is always an incentive for coalitions to form.

Definition: Partition Q is a **refinement** of P if $\forall R \in Q,\ \exists S \in P$ s.t. $R \subseteq S$.

Definition: A PFF game v is **partition monotonic** if, whenever $P, Q \in PT(S)$ for some $S \subseteq N$, and Q is a refinement of P, we have $v(S;Q) \geq v(S;P)$.

Roughly speaking, partition monotonicity says that coalition S will benefit when there is less collaboration in N\S.

We can now show that these two conditions are sufficient for symmetric games to have $\overline{D}$ nonempty.

Theorem II: If a PFF game v is symmetric, superadditive, and partition monotonic, then $\overline{D}$ is nonempty.

Proof II:

Let $\{i \in S\}$ denote the partition of N in which all the members of S work individually, and all the members of N\S work together. Let n and s denote the cardinality of N and S respectively. We make the following observations:

- Superadditivity and symmetry give that $v(i;\ \{j \in N\}) \leq \frac{1}{n} v(N;N)\quad \forall i \in N$.

- Since v is symmetric, an allocation that satisfies ETP seems appropriate. So we wish to show that $\hat{x} \in \overline{E}$ s.t. $\hat{x}(i) = \frac{1}{n} v(N;N)$ is contained in $\overline{D}$; i.e. that $\hat{x}(S)$ is reasonable for all $S \subseteq N$. If not, then for some $S \subseteq N$, the following hold:

1. $\exists P \in PT(S)$ s.t. $v(S;P) < \hat{x}(S) = \frac{s}{n} v(N;N)$.
2. $\exists x \in \overline{E}$ satisfying:

   (i) $x(R) \leq v(R;P)\quad \forall S \neq R \in P$ and

(ii) $x(T) > \max[v(S \cup T;Q) - \hat{x}(S):\ \ Q \in PT(S \cup T)\ ]\quad \forall \varnothing \neq T \subseteq N \setminus S$.

Notice that the reasonableness of $\hat{x}(N)$ is trivial since (1) is violated. Also, since v is partition monotonic, we have that P = {S, N\S} satisfies (1) for any $S \subseteq N$. This is the partition that assigns S the minimum possible amount, and if there exists another $P \in PT(S)$ which assigns S the minimum possible amount, we may still use P because:

(a) (2i) is automatically satisfied since

$$x(R) \leq v(R;P')\ \ \forall R \in P' \quad \Rightarrow \quad x(N \setminus S) \leq \sum_{S \neq R \in P'} v(R;P') \leq v(N \setminus S;P)$$

(The last inequality holds by superadditivity.)

(b) The $T \subseteq N \setminus S$ considered in (2ii) do not depend on which partition is chosen.

Furthermore, for each $T \subseteq N \setminus S$ we know which $Q \in PT(S \cup T)$ maximizes the right hand side in (2ii). It is $Q = \{i \in N \setminus (S \cup T)\}$.

Notice also that in (2) we can use $x \in \overline{E}$ such that $x(j) = \frac{1}{n-s} v(N \setminus S;\ \{S,\ N \setminus S\})$ $\forall j \in N \setminus S$ (we are no longer considering S=N), and we would have equality hold in (2i). Suppose the actual $x \in \overline{E}$ gives $x(j) > \frac{1}{n-s} v(N \setminus S;\ \{S,\ N \setminus S\})$ for some $j \in N \setminus S$, so that $x(k) < \frac{1}{n-s} v(N \setminus S;\ \{S,\ N \setminus S\})$ for some $k \in N \setminus S$. Then we would have in (2ii) that $x(j) > v(S \cup \{j\};\ \{j \in N \setminus (S \cup T)\}) - \frac{s}{n} v(N;N)$ holds, but since $v(S \cup \{j\};\ \{j \in N \setminus (S \cup T)\}) = v(S \cup \{k\};\ \{j \in N \setminus (S \cup T)\})$, we have that $x(j) = x(k) > v(S \cup \{j\};\ \{j \in N \setminus (S \cup T)\}) - \frac{s}{n} v(N;N)$ also holds, and so the $x \in \overline{E}$ we chose above still works. We summarize our observations:

If $\hat{x}(S)$ is unreasonable for some $S \subset N$, then

1'. P={S, N\S} satisfies $v(S;\ \{S,\ N \setminus S\}) < \frac{s}{n} v(N;N)$.

2'. $\exists x \in \overline{E}$ s.t. $\forall j \in N \setminus S \quad x(j) = \frac{1}{n-s} v(N \setminus S;\ \{S,\ N \setminus S\})$ satisfying:

(i)' $x(N \setminus S) = v(N \setminus S;\ \{S,\ N \setminus S\})$, and $\forall \varnothing \neq T \subseteq N \setminus S$, t=|T|

(ii)' $x(T) = \frac{t}{n-s} v(N \setminus S;\ \{S,\ N \setminus S\}) > v(S \cup T;\ \{j \in N \setminus (S \cup T)\}) - \frac{s}{n} v(N;N)$, Notice that, in particular, when T = N\S we get in **(2ii)'**:

$$x(N \setminus S) = v(N \setminus S;\, \{S,\ N \setminus S\}) > v(N;N) - \tfrac{s}{n} v(N;N) = \tfrac{n-s}{n} v(N;N);$$

that is, **(*)** $v(N \setminus S;\, \{S,\ N \setminus S\}) > \tfrac{n-s}{n} v(N;N)$.

<u>Case 1</u>:

Choose $T \subseteq N \setminus S$ such that |T| = t = n-2s > 0. Then we have that in **(2ii)'** above

$|S \cup T| = s + t = s + n - 2s = n - s$, and

$|\{j \in N \setminus (S \cup T)\}| = n - (n-s) = s$.

So, by symmetry $v(S \cup T;\ \{j \in N \setminus S\}) = v(N \setminus S;\ \{i \in S\})$ and **(2ii)'** can be written:

$$x(T) > v(N \setminus S;\ \{i \in S\}) - \tfrac{s}{n} v(N;N) \geq v(N \setminus S;\ \{S,\ N \setminus S\}) - \tfrac{s}{n} v(N;N).$$

(The last inequality holds by partition monotonicity.) Equivalently,

$$\tfrac{t}{n-s} v(N \setminus S;\ \{S,\ N \setminus S\}) > v(N \setminus S;\ \{S,\ N \setminus S\}) - \tfrac{s}{n} v(N;N)$$

$$\Leftrightarrow \quad \tfrac{s}{n} v(N;N) > \tfrac{n-s}{n-s} - \tfrac{(n-2s)}{n-s}[v(N \setminus S;\ \{S,\ N \setminus S\})]$$

$$\Leftrightarrow \quad \tfrac{s}{n} v(N;N) > \tfrac{s}{n-s} v(N \setminus S;\ \{S,\ N \setminus S\})$$

$$\Leftrightarrow \quad \tfrac{n-s}{n} v(N;N) > v(N \setminus S;\ \{S,\ N \setminus S\})$$

which contradicts **(*)** above.

<u>Case 2</u>:

Suppose that n-2s≤0. Then choose T such that (n-s) | (s+t). Notice that this can be done because:

- n-s ≤ s < n ⇒ n-s < s+t ≤ n for t = 1, 2, . . . , n-s
- n-s < n ⇒ $\exists k, r \in Z$ s.t. $n = k(n-s) + r$ with $0 \leq r < n-s$
- letting t = n-r-s = k(n-s) - s so that t+s = k(n-s), we get

  1≤ t = n-s-r ≤ n-s, which is always possible.

With T such that s+t = k(n-s), and by superadditivity, we get

$$v(S \cup T;\ \{j \in N \setminus (S \cup T)\}) \geq k\, v(N \setminus S;\ Q)$$

where Q is a partition consisting of k sets of size (n-s) each, and n-(s+t) players working individually. So **(2ii)'** above can be written:

$$x(T) > k\, v(N \setminus S;\, Q) - \tfrac{s}{n} v(N;N)$$

Notice that Q is a refinement of {S, N\S}, and so by partition monotonicity we have:

$$\tfrac{t}{n-s} v(N \setminus S;\, Q) \geq \tfrac{t}{n-s} v(N \setminus S;\, \{S,\ N \setminus S\})$$

So **(2ii)'** now becomes

$$\tfrac{t}{n-s} v(N \setminus S;\, Q) > k\, v(N \setminus S;\, Q) - \tfrac{s}{n} v(N;N)$$

$$\Leftrightarrow \quad \tfrac{t}{n-s} v(N \setminus S;\, Q) > \tfrac{t+s}{n-s} v(N \setminus S;\, Q) - \tfrac{s}{n} v(N;N)$$

$$\Leftrightarrow \quad \tfrac{s}{n} v(N;N) > \tfrac{t+s}{n-s} - \tfrac{t}{n-s}[\, v(N \setminus S;\, Q)\,]$$

$$\Leftrightarrow \quad \tfrac{s}{n} v(N;N) > \tfrac{s}{n-s} v(N \setminus S;\, Q)$$

$$\Leftrightarrow \quad \tfrac{1}{n} v(N;N) > \tfrac{1}{n-s} v(N \setminus S;\, Q)$$

$$\Leftrightarrow \quad \tfrac{n-s}{n} v(N;N) > v(N \setminus S;\, Q)$$

which again contradicts **(*)** above, and Theorem II is now established.

Conclusion:

We have defined a class of reasonable outcomes for PFF games, and we determined the nonemptiness of this set for symmetric games. A more general result is desired, however. It is not clear that Theorem II is true for non-symmetric PFF games: The nonemptiness of the $\overline{D}$ set for such games might require additional game properties. Variations in the definition of $\overline{D}$ might prove more fruitful, and other concepts of reasonableness should be studied as well as their relation to concepts of rationality.

As for the μ-value, we still need to determine a computation method. To do this, we might want to consider how balanced sets and properties of consistency and covariance should be defined for values of PFF games. The hope is that this value will be easier to compute than the extension of the Shapley value for PFF games.

References

Luce, R. D. and Raiffa, H. "Reasonable Outcomes and Value" in Games and Decisions, (New York: John Wiley & Sons, 1957), pp. 237-253.

Housman, David. "Allocation Methods For Cooperative Games: An Axiomatic Approach," Drew University, NJ, 1991.

Grotte, Jeffrey H. Computation of and Observations on the Nucleolus, the Normalized Nucleolus, and the Central Games, (M.S. dissertation, Department of Mathematics, Cornell University, 1970).

Merki, Stephanie. "Values on Partition Function Form Games," REU Cooperative Game Theory, Drew University, NJ, 1991.

Housman, David. Personal Communication, REU Cooperative Game Theory, Drew University, NJ, 1992.