Reasonable Allocations for Cooperative Games

By Luz E. Pinzon
Henry Rutgers Scholar, 1992-93

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Introduction

In 1944 J. von Neumann and O. Morgenstern published the now classic book *Theory of Games and Economic Behavior* which laid the foundation for the modern mathematical approach to situations of conflict and cooperation. Game theory aims to model situations in which the interaction of two, or more, individuals (players) leads to potential payoffs over which each player has his own preferences. In this paper, we shall be concerned only with cooperative games and one specific model for them, mainly the *coalitional* model. In these circumstances, the players have complete freedom of pre-play communication to make joint, binding agreements. Thus, coalitions will be formed and they will, as a whole, strive to achieve as large a total payoff as possible. This payoff is a function of the coalition, and the problem arises, then, as to how the total payoff gained by each coalition should be divided among its members. Consider, for instance, the following scenario:

Three adjacent cities are required by the EPA to improve the quality of the air by some measure. Each city could work individually and invest a certain amount in the necessary technology to comply with the EPA's new standards. However, each city realizes that joining its efforts with at least one other city results in attractive savings. That is, there are incentives for each city to cooperate with others, to form "coalitions." Ultimately, they realize that the most savings are achieved if all three cities join and cooperate in one "grand coalition." Now, it must be decided how these savings are to be distributed among the three cities.

A cooperative game in coalition function form (CFF) consists of a set of players, the three cities for example, and a "worth" function which assigns a real value or "savings amount" to each coalition of players. It may be the case that the worth assigned to each coalition also depends on how the rest of the players cooperate. For example, consider an oligopoly where a single product is manufactured by a finite (usually small) number of firms. The profits of one firm, or a group of firms, is likely to be influenced by how the remaining firms collude. This situation can be modeled using cooperative games in partition function form (PFF), where the "partition" to which a coalition belongs determines the worth of the coalition. For either type of game, however, the objective is the same: to allocate the worth of the grand coalition among all players.

The payoff to be awarded to each player is determined according to certain solution concepts. These, in turn, are based on specific interpretations of the fairness of potential payoffs. And so, in an attempt to choose the best allocation for any given game, concepts of "reasonableness" often arise. The problem of developing an appropriate theory of reasonable allocations becomes, naturally, an important one, and this is precisely the motivation for this work.

We begin the paper with some of the basic concepts of cooperative games that will be relevant to the theory of reasonableness. Also, some of the most common properties that are desirable of any "reasonable" allocation, or outcome, are studied in this first section. We then examine in detail Milnor's notion of reasonable *demands* in section II, and we use a linear programming approach to determine the maximum reasonable payoff that any coalition can demand in a given game. The extreme points of this linear program are characterized, and further simplifications are made when the game is assumed to be "balanced." These observations become helpful in section III where we investigate Milnor's class D of reasonable outcomes, and we find that this set coincides precisely with the *core*, or the set of allocations that are "group rational," for balanced games. The concept of *strong* ε*-cores* is another tool used in this section that gives us a sufficient condition for D to be empty, which can indeed be the case as shown with an example. Finally, in section IV, Milnor's concept of reasonable demands is extended to PFF games, and the nonemptiness of the analogous class $\overline{D}$ is generalized for symmetric PFF games.

I. Basic Notions

Cooperative Games

A cooperative game consists of a pair (N, v), where N is the set of players, and v is a real valued function on the subsets of N; that is, $v: 2^N \rightarrow \mathbf{R}$. We usually write N={1, 2, 3, ..., n}. The function v is called the **characteristic function** of the game, and v(S) represents the "worth" of **coalition** S, where $S \subseteq N$. By convention $v(\emptyset) = 0$. We will refer to these games as Coalition Function Form (CFF) games, or simply games whenever the form is understood.

A CFF game (N, v) is **superadditive** if there are incentives for coalitions to form, or, formally, if $v(S \cup T) \geq v(S) + v(T)$ for all coalitions $S, T \subseteq N$ satisfying $S \cap T = \emptyset$. *In this paper we will restrict ourselves to superadditive games.*

Allocations

An allocation, or an **outcome**, for the game (N, v) is a vector $\mathbf{x} \in \mathbf{R}^n$, where the i^{th} coordinate of **x** represents the **payoff** to player i. The payoff to a coalition $S \subseteq N$ is denoted **x**(S), where $\mathbf{x}(S) = \sum_{i \in S} \mathbf{x}_i$.

An allocation is said to be **efficient** if **x**(N) = v(N); that is, if the maximum worth is distributed among all the players. Efficient allocations are also known as **pre-imputations**, and the set of all pre-imputations will be denoted **E**. That is,

$$\mathbf{E} = \left\{\mathbf{x} \in \mathbf{R}^n \colon \sum_{i \in N} \mathbf{x}_i = v(N)\right\}.$$

Efficiency is clearly a desirable property for any allocation to have, but additional requirements are often sought. For instance, it might be argued that the payoff to each player should be at least the amount he can realize by his own right. Then, we want to consider only those allocations that are "individually rational," or the set of all **imputations**, which is given by:

$$IR(v) = \{x \in E: \ x(i) \geq v(i)\}$$ [1]

The notion of rationality can also be extended to groups, and we define the set of all efficient allocations that are "group rational" to be the **core of the game**, which we denote C(v), and we write,

$$C(v) = \{x \in E: \ x(S) \geq v(S) \ \ \forall S \subseteq N\}.$$

For the games studied in this paper, the core consists of a closed, convex polyhedron in the space of payoff vectors.

Balanced Games

An important characterization of the core will be used here which requires the study of balanced collections and balanced games. These notions were introduced by Shapley [1967], who studied the relationship between the balanced collections of a game and the conditions that determine when the game has an empty core.

Definition: Consider a collection $C = \{S_1, S_2, \ldots, S_m\}$ of distinct nonempty subsets of N. This collection is said to be **balanced over** N if there exist positive real numbers $\lambda_1, \lambda_2, \ldots, \lambda_m$, called balancing **weights**, that satisfy for each $i \in N$, $\sum_{j: i \in S_j} \lambda_j = 1$. The vector $(\lambda_1, \lambda_2, \ldots, \lambda_m)$ is called the **balancing vector**.

Notice that the weights are equal to 1 if and only if the balanced collection C over N is a partition of N, and so balanced collections may be regarded as generalized partitions. For example, the collection { {1,2}, {1,3}, {2,3} } is balanced over N = {1, 2, 3} with balancing vector $(\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$, and the collection { {1,2}, {1,3}, {1,4}, {2,3,4} } is balanced over N = {1, 2, 3, 4} with balancing vector $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \frac{2}{3})$.

Definition: A game (N, v) is said to be a **balanced game** if, for any balanced collection $C = \{S_1, S_2, \ldots, S_m\}$ over N with corresponding balancing vector $(\lambda_1, \lambda_2, \ldots, \lambda_m)$, the "balanced inequality" $\lambda_1 v(S_1) + \lambda_2 v(S_2) + \ldots + \lambda_m v(S_m) \leq v(N)$ is satisfied.

The following results are proved in Shapley [1967]:

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[1] For simplicity, we ignore commas and brackets when listing the members of a coalition S of N.

Theorem 1: A game (N, v) has nonempty core C(v) if, and only if, (N, v) is a balanced game. ◊

It is a fact that every balanced collection is the union of "minimal" balanced collections. **Minimal balanced collections** are those which include no other balanced collection and have unique balancing weights. So the theorem above also holds for **minimal balanced games**; that is, a game has nonempty core if and only if, for any minimal balanced collection $C = \{S_1, S_2, \ldots, S_m\}$ with corresponding balancing vector$(\lambda_1, \lambda_2, \ldots, \lambda_m)$, we have $\lambda_1 v(S_1) + \lambda_2 v(S_2) + \ldots + \lambda_m v(S_m) \leq v(N)$.

Notice that if a game is superadditive, the balanced inequalities are automatically satisfied for the balanced collections that form partitions of N. In a three-player, superadditive game with $N = \{1, 2, 3\}$, for instance, the collection $\{ \{1,2\}, \{1,3\}, \{2,3\} \}$ is minimal balanced, and in fact, it is the only minimal balanced collection which does not form a partition of N. Thus, the only condition for a superadditive game with n = 3 to have a nonempty core is that $v(12) + v(13) + v(23) \leq 2v(123)$.

Symmetric Games

A game (N, v) is **symmetric** if v(S) = v(T) for all coalitions $S, T \subseteq N$ such that $|S| = |T|$.

Notation: If (N, v) is symmetric, define $v_s = v(S)$ for all $S \subseteq N$ with $|S| = s$.

Consider a symmetric game with n = 3. Then the inequality needed to have a nonempty core becomes $3v_2 \leq 2v_3$, or $\frac{1}{2}v_2 \leq \frac{1}{3}v_3$. Superadditivity also gives $v_1 \leq \frac{1}{2}v_2$, so that if we regard $\frac{1}{k}v_k$ to be the "per-capita worth" that a player receives in a coalition of size k, we see this is largest when k = 3 = n This is, indeed, the case in general:

Proposition 1: If (N, v) is a symmetric game, then C(v) is nonempty if and only if $\frac{1}{n}v_n \geq \frac{1}{s}v_s$ for all s = 1, 2, . . . , n.

<u>Proof</u>:

If $\frac{1}{n}v_n \geq \frac{1}{s}v_s$ for all s = 1, 2, . . . , n then $x = (\frac{1}{n}v_n, \frac{1}{n}v_n, \ldots, \frac{1}{n}v_n) \in C(v)$ since $x(S) = \frac{s}{n}v_n \geq v_s = v(S)$ for each $S \subseteq N$. If C(v) is nonempty, then for each $x \in C(v)$, and for each s = 1, 2, . . . , n, we have $v_s \leq \min\{x(S): |S| = s\} \leq \frac{s}{n}v_n$. So $\frac{1}{n}v_n \geq \frac{1}{s}v_s$. ◊

II. Reasonableness

John Milnor [1952] defined lower and upper bounds for the payoffs that each coalition should receive in any "reasonable" play of a given CFF game. Of particular interest in this paper will be the notion of "reasonable demands."

Definition: Let d(S) represent the payoff that coalition S demands in a game (N, v) with $S \subseteq N$. Then d(S) is **unreasonable** for S if there exists an allocation $\mathbf{x} \in \mathbf{E}$ so that:

[i] $x(N\backslash S) \leq v(N\backslash S)$, and

[ii] $x(R) > v(S \cup R) - d(S)$, for each $R \subseteq N \setminus S$.

That is, d(S) is unreasonable for coalition S if its complement can "prevent" it from getting d(S) by enforcing some allocation **x** which is "feasible" (condition [i]), and such that no subset of N\S can be induced to join with S (condition [ii]), knowing S will keep d(S).

Definition: A payoff for coalition S is **reasonable** if it is not unreasonable.

Notice that d(N) is reasonable if and only if $d(N) \leq v(N)$.

Proposition 2: A demand d(N\i) is reasonable for N\i if and only if $d(N\backslash i) \leq v(N) - v(i)$.

<u>Proof</u>:

To say that d(N\i) is unreasonable for N\{i} means there exists $\mathbf{x} \in \mathbf{E}$ so that:

[i] $x(i) \leq v(i)$, and

[ii] $x(i) > v(N) - d(N\backslash i)$.

Conditions [i] and [ii] hold if and only if $v(i) > v(N) - d(N\backslash i)$, or equivalently, $d(N\backslash i) > v(N) - v(i)$. ◊

In general, the **maximum reasonable demand** for a coalition $S \subseteq N$, denoted $\delta(S)$, can be obtained by solving the linear program:

$$\delta(S) = \min \ \delta$$
$$\text{s. t.} \quad x(N \setminus S) \leq v(N \setminus S) \qquad \text{(LP1)}$$
$$\delta + x(R) \geq v(S \cup R), \quad R \subseteq N \setminus S$$

For our purposes in this paper, the dual formulation of $\delta(S)$ will be of greater use. We can write the dual of the linear program above as follows:

$$\delta(S) = \max \ -\lambda_0 v(N \setminus S) + \sum_{R \subseteq N \setminus S} \lambda_R v(S \cup R)$$
$$\text{s. t.} \quad \sum_{R \subseteq N \setminus S} \lambda_R = 1$$
$$\sum_{\substack{R \ni i \\ R \subseteq N \setminus S}} \lambda_R = \lambda_0, \qquad i \in N \setminus S, \qquad \text{(LP2)}$$
$$\lambda_0 \geq 0, \ \ \lambda_R \geq 0, \qquad R \subseteq N \setminus S.$$

Proposition 3: If (N, v) is any game and $S \subseteq N$, then $\delta(S) \geq v(S)$.

Proof:

Notice that v(N) - v(N\S) is a feasible value for $\delta(S)$ in LP2 if we let $\lambda_0 = \lambda_{N \setminus S} = 1$, and $\lambda_R = 0$ otherwise. This gives that $\delta(S) \geq v(N) - v(N \setminus S) \geq v(S)$, where the last inequality holds by superadditivity. ◊

This says that we need not consider $R = \emptyset \subseteq N \setminus S$ when determining $\delta(S)$ for any $S \subseteq N$.

Example 1

Consider a game (N, v) with N = {1, 2, 3}. To determine $\delta(i)$ for any singleton $i \in N$, suppose that δ is unreasonable for player i. Then there exists $x \in E$ such that :

[i] $x(jk) \leq v(jk)$;

[ii] $x(j) > v(ij) - \delta$
$x(k) > v(ik) - \delta$
$x(jk) > v(ijk) - \delta$

Adding the first two inequalities in [ii] yield

$$[ii] \Rightarrow \begin{cases} x(jk) > v(ij) + v(ik) - 2\delta \\ x(jk) > v(ijk) - \delta \end{cases}$$

which can now be combined with [i] to require

$$\left.\begin{array}{l} v(jk) > v(ij) + v(ik) - 2\delta \\ v(jk) > v(ijk) - \delta \end{array}\right\} \Leftrightarrow \left\{\begin{array}{l} \delta > \frac{1}{2}v(ij) + \frac{1}{2}v(ik) - \frac{1}{2}v(jk) \\ \delta > v(ijk) - v(jk) \end{array}\right.$$

So if δ is unreasonable for player $i \in N$, then $\delta > \max\{\ \frac{1}{2}v(ij) + \frac{1}{2}v(ik) - \frac{1}{2}v(jk),\ v(ijk) - v(jk)\ \}$.

We will show now that these values occur precisely at the basic solutions of the LP2, so that $\delta(i) = \max\{\ \frac{1}{2}v(ij) + \frac{1}{2}v(ik) - \frac{1}{2}v(jk),\ \ v(ijk) - v(jk)\ \}$. We write:

$$\delta(i) = \max\ \ -\lambda_0 v(jk) + \lambda_j v(ij) + \lambda_k v(ik) + \lambda_{jk} v(ijk)$$

$$\begin{array}{ll} \text{s. t.} & \lambda_j + \lambda_k + \lambda_{jk} = 1 \\ & \lambda_j \qquad + \lambda_{jk} = \lambda_0 \\ & \qquad \lambda_k + \lambda_{jk} = \lambda_0 \\ & \lambda_0, \lambda_j, \lambda_k, \lambda_{jk} \geq 0 \end{array}$$

If $\lambda_j = 0$, we obtain $\lambda_k = 0$ and $\lambda_{jk} = \lambda_0 = 1$, which gives $\delta(i) = v(ijk) - v(jk)$. If we let $\lambda_k = 0$, we obtain the same result. Finally, letting $\lambda_{jk} = 0$ gives $\lambda_j = \lambda_k = \lambda_0 = \frac{1}{2}$ and $\delta(i) = \frac{1}{2}v(ij) + \frac{1}{2}v(ik) - \frac{1}{2}v(jk)$. So $\delta(i)$ is given by the maximum of these two values. Now, since any pair of players in N is of the form $N\backslash\{i\}$, then $\delta(jk) = v(ijk) - v(i)$ as shown in Proposition 2 for any $i \in N$.

Suppose that $\frac{1}{2}v(ij) + \frac{1}{2}v(ik) - \frac{1}{2}v(jk) \leq v(ijk) - v(jk)$, or equivalently that $v(ij) + v(ik) + v(jk) \leq 2v(ijk)$. The same inequality occurs for any $i \in N$, so that each $\delta(i)$ is given by the same feasible solution, namely $v(ijk) - v(jk)$. Notice that this inequality is precisely the requirement for the three-player game to have a nonempty core. In general, $\delta(S)$ can be easily determined for any $S \subseteq N$ whenever the core of the game is nonempty:

Proposition 4: If (N, v) is a CFF game whose core $C(v)$ is nonempty, then for each $S \subseteq N$, $\delta(S) = v(N) - v(N \setminus S)$.

<u>Proof</u>:

Suppose $C(v) \neq \varnothing$. Then all balanced inequalities of the form

(*) $\quad \lambda_1 v(S_1) + \lambda_2 v(S_2) + \ ...\ + \lambda_k v(S_k) \leq v(N)$, are satisfied with all $\lambda_j \geq 0$.

From Proposition 3 we know $\delta(S) \geq v(N) - v(N \setminus S)$. We now claim,

$\delta(S) \leq v(N) - v(N \setminus S)$, which is equivalent to:

$$-\lambda_0 v(N \setminus S) + \sum_{R \subseteq N \setminus S} \lambda_R v(S \cup R) \leq v(N) - v(N \setminus S)$$

$$\text{or} \quad (1-\lambda_0) v(N \setminus S) + \sum_{R \subseteq N \setminus S} \lambda_R v(S \cup R) \leq v(N)$$

for all feasible solutions λ to LP2.

Since $0 \leq \lambda_0 \leq 1$, then $(-\lambda_0 + 1) \geq 0$. Also all $\lambda_R \geq 0$.

Hence, if $i \in N \setminus S$, we have $(1-\lambda_0) + \sum_{R \ni i} \lambda_R = (1-\lambda_0) + \lambda_0 = 1$. If, on the other hand $i \in S$, then $\sum_{R \subseteq N \setminus S} \lambda_R = 1$. Thus, our claim is in the form of balanced inequalities which now hold by (*). ◊

Observe that, whenever the core is empty in a three-player game, each $\delta(i)$ is given by $\frac{1}{2}v(ij) + \frac{1}{2}v(ik) - \frac{1}{2}v(jk)$. The general case is not as easily resolved; however, we are able to characterize the extreme points of LP2. Notice that the feasible solutions for LP2 are the same feasible solutions for the following linear program:

$$\delta(S) + v(N \setminus S) = \max \ (1-\lambda_0) v(N \setminus S) + \sum_{R \subseteq N \setminus S} \lambda_R v(S \cup R)$$

$$\begin{aligned} \text{s. t.} \quad & \sum_{R \subseteq N \setminus S} \lambda_R = 1 \\ & \sum_{\substack{R \ni i \\ R \subseteq N \setminus S}} \lambda_R = \lambda_0, \qquad i \in N \setminus S, \qquad \text{(LP2')} \\ & \lambda_0 \geq 0, \ \lambda_R \geq 0, \quad R \subseteq N \setminus S. \end{aligned}$$

This LP2' can be thought of as a "restricted" form of another linear program, mainly:

$$\begin{aligned} \max \ & \sum_{S \subseteq N} \lambda_S v(S) \\ \text{s.t.} \ & \sum_{S:\, S \ni i} \lambda_S = 1 \quad \text{for all } i \in N \qquad \text{(LP)} \\ & \lambda_S \geq 0 \qquad \text{for all } S \subseteq N \end{aligned}$$

Owen [1982] proves that the extreme points of LP are precisely the vectors of the minimal balanced collections over N, and in a similar way we can show the following assertion.

Proposition 5: The extreme points of LP2' are the balancing vectors of the minimal balanced collections over N which contain only supersets of S and possibly N\S.

Proof:

First, notice that such vectors are indeed feasible solutions of LP2'. Now, suppose the vector $\lambda = (\lambda_T)_{T \subseteq N}$, where $T \subseteq N$ is of the required form, satisfies the restrictions of LP2'. Then it is a balancing vector for the collection $C_1 = \{T: \lambda_T > 0\}$. Suppose C_1 is not minimal. We claim λ is not an extreme point of LP2'. Let C_2 be a balanced proper subcollection of C_1 with balancing vector $\mu = (\mu_T)_{T \subseteq N}$. We have that $\mu_T > 0$ only if $\lambda_T > 0$, and so for small values of t, define

$$\begin{matrix} \gamma_T = (1-t)\lambda_T + t\mu_T \\ \gamma_0 = (1-t)\lambda_0 + t\mu_0 \end{matrix} \quad \text{and} \quad \begin{matrix} \gamma'_T = (1+t)\lambda_T - t\mu_T \\ \gamma'_0 = (1+t)\lambda_0 - t\mu_0 \end{matrix} \quad \text{for appropriate } T \subseteq N$$

Then γ and γ' satisfy the restrictions of LP2', and since $\gamma_T < \gamma'_T$ for any $T \in C_1 \setminus C_2$, then $\gamma \neq \gamma'$. But this gives that $\lambda = \frac{1}{2}(\gamma + \gamma')$, and so λ is not an extreme point of LP2'.

Conversely, suppose that C is a minimal balanced collection of the required form. If its corresponding balancing vector, λ, is not an extreme point, then there exist distinct γ and γ' satisfying the restrictions of LP2' and such that $\lambda = \frac{1}{2}(\gamma + \gamma')$. Because of the nonnegativity constraints of LP2', we must have $\gamma_T = \gamma'_T = 0$ whenever $\lambda_T = 0$, and so γ and γ' will be balancing vectors for C_1 and C_2, respectively, where

$$C_1 = \{T: \gamma_T > 0\} \qquad \text{and} \qquad C_2 = \{T: \gamma'_T > 0\}$$

are both subcollections of C. Since C is minimal, then we must have $C = C_1 = C_2$, and by the uniqueness of λ, we also have $\lambda = \gamma = \gamma'$. This contradiction now gives that λ is an extreme point of LP2'. ◊

Example 2

For a game (N, v) with N = {1, 2, 3, 4} and $C(v) = \emptyset$, all the $\delta(S)$ are of the following form:

$$\delta(i) = \max \begin{Bmatrix} \frac{1}{3}[v(ij) + v(ik) + v(il) - v(jkl)]; \\ \frac{1}{2}[v(il) + v(ijk) - v(jkl)]; \\ \frac{1}{2}[v(ik) + v(ijl) - v(jkl)]; \\ \frac{1}{2}[v(ij) + v(ikl) - v(jkl)]; \\ \frac{1}{3}[v(ijk) + v(ijl) + v(ikl) - 2v(jkl)]; \\ v(ijkl) - v(jkl) \end{Bmatrix} \text{ for each } i \in N;$$

$$\delta(ij) = \max \begin{Bmatrix} \frac{1}{2}[v(ijk) + v(ijl) - v(kl)]; \\ v(ijkl) - v(kl) \end{Bmatrix} \text{ for any pair of players } i, j.$$

$$\delta(ijk) = v(ijkl) - v(l) \quad \text{for any three players } i, j, k.$$

To show this, suppose first that δ is unreasonable for player i. This means that there exists $\mathbf{x} \in E$ satisfying all of the following:

$$\left.\begin{matrix} x(jkl) \le v(jkl) \\ x(j) > v(ij) - \delta \\ x(k) > v(ik) - \delta \\ x(l) > v(il) - \delta \\ x(jk) > v(ijk) - \delta \\ x(jl) > v(ijl) - \delta \\ x(kl) > v(ikl) - \delta \\ x(jkl) > v(ijkl) - \delta \end{matrix}\right\} \Rightarrow \begin{cases} x(jkl) \le v(jkl) \\ x(jkl) > v(ij) + v(ik) + v(il) - 3\delta \\ x(jkl) > v(il) + v(ijk) - 2\delta \\ x(jkl) > v(ik) + v(ijl) - 2\delta \\ x(jkl) > v(ij) + v(ikl) - 2\delta \\ 2x(jkl) > v(ijk) + v(ijl) + v(ikl) - 3\delta \\ x(jkl) > v(ijkl) - \delta \end{cases}$$

So we must have,

$$\left.\begin{matrix} v(jkl) > v(ij) + v(ik) + v(il) - 3\delta \\ v(jkl) > v(il) + v(ijk) - 2\delta \\ v(jkl) > v(ik) + v(ijl) - 2\delta \\ v(jkl) > v(ij) + v(ikl) - 2\delta \\ 2v(jkl) > v(ijk) + v(ijl) + v(ikl) - 3\delta \\ v(jkl) > v(ijkl) - \delta \end{matrix}\right\} \Leftrightarrow \begin{cases} \delta > \frac{1}{3}[v(ij) + v(ik) + v(il) - v(jkl)] \\ \delta > \frac{1}{2}[v(il) + v(ijk) - v(jkl)] \\ \delta > \frac{1}{2}[v(ik) + v(ijl) - v(jkl)] \\ \delta > \frac{1}{2}[v(ij) + v(ikl) - v(jkl)] \\ \delta > \frac{1}{3}[v(ijk) + v(ijl) + v(ikl) - 2v(jkl)] \\ \delta > v(ijkl) - v(jkl) \end{cases}$$

We now claim that $\delta(i)$ is given by the maximum of the values on the right-hand side. This is justified by considering the extreme points of LP2' for $S = \{i\}$. The minimal balanced collections for a four-player game (which contain only supersets of {i} and no proper subsets of {jkl}) are, with their respective balancing vectors: [2]

$C_m = \{T\text{: T as appropriate}\}$	λ
$\{ij, ik, il, jkl\}$	$(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \frac{2}{3})$
$\{ij, ikl, jkl\}$	$(\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$
$\{il, ijk, jkl\}$	$(\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$
$\{ik, ijl, jkl\}$	$(\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$
$\{ijk, ijl, ikl, jkl\}$	$(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3})$
$\{i, jkl\}$	(1, 1)
$\{ijkl\}$	(1)

It can be easily checked that we obtain the values on the right-hand side of δ by computing, for each C_m, $\left[\sum_{T\in C_m} \lambda_T v(T)\right] - v(jkl)$. Notice that the collection {i, jkl} gives us the feasible solution v(i), which we showed was redundant in Proposition 3.

Thus, $\delta(i) = \max\limits_{C_m} \sum_{T\in C_m} \lambda_T v(T) - v(jkl)$.

Suppose now that δ is unreasonable for the pair {ij}. This means that there is some $x \in E$ for which

$$\left.\begin{array}{l} x(kl) \le v(kl) \\ x(k) > v(ijk) - \delta \\ x(l) > v(ijl) - \delta \\ x(kl) > v(ijkl) - \delta \end{array}\right\} \Rightarrow \begin{cases} x(kl) \le v(kl) \\ x(kl) > v(ijk) + v(ijl) - 2\delta \\ x(kl) > v(ijkl) - \delta \end{cases}$$

And we must have

$$\left.\begin{array}{l} v(kl) > v(ijk) + v(ijl) - 2\delta \\ v(kl) > v(ijkl) - \delta \end{array}\right\} \Leftrightarrow \begin{cases} \delta > \frac{1}{2}[v(ijk) + v(ijl) - v(kl)] \\ \delta > v(ijkl) - v(kl) \end{cases}$$

Again, we can justify that $\delta(ij)$ is given by the maximum of the two values above, for the only minimal balanced collections (containing only supersets of {ij}, and possibly {kl}) are:

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2 These can be found in Owen [1982], p. 162.

{ijk, ijl, kl}	$(\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$
{ij, kl}	(1, 1)
{ijkl}	(1)

These are the extreme points of LP2' for S = {ij}.

For any triple {ijk}, δ(ijk) can be found using Proposition 2.

Definition: If (N, v) is a symmetric game, let δ_s denote $\delta(S)$ for all $S \subseteq N$ with $|S| = s$.

Proposition 6: If (N, v) is a symmetric game, then, for each s = 1, . . . , n, $\delta_s = \max\left\{ v_{r+s} - \frac{r}{n-s} v_{n-s} \quad : r = 0, 1, \ldots, n-s \right\}$.

Proof:
Let $\gamma_r = \sum_{|R|=r} \frac{\lambda_R}{\binom{n-s}{r}}$ for any feasible solution λ of LP2. Then:

$$\delta_s = \max -\lambda_0 v_{n-s} + \sum_{r=0}^{n-s} \binom{n-s}{r} \gamma_r v_{r+s}$$

$$\text{s. t. } \sum_{r=0}^{n-s} \binom{n-s}{r} \gamma_r = 1$$

$$\sum_{r=1}^{n-s} \binom{n-s-1}{r-1} \gamma_r = \lambda_0$$

$$\lambda_0 \geq 0, \quad \gamma_r \geq 0, \quad r = 0, 1, \ldots, n\text{-}s$$

Substituting for λ_0 we obtain,

$$\delta_s = \max \gamma_0 v_s + \sum_{r=1}^{n-s} \gamma_r \left[\binom{n-s}{r} v_{r+s} - \binom{n-s-1}{r-1} v_{n-s} \right]$$

$$\text{s. t. } \sum_{r=0}^{n-s} \binom{n-s}{r} \gamma_r = 1;$$

$$\gamma_r \geq 0, \quad r = 0, 1, \ldots, n\text{-}s.$$

Using $\hat{\gamma}_r = \binom{n-s}{r} \gamma_r$ and the fact that $\binom{n-s-1}{r-1} = \frac{r}{n-s} \binom{n-s}{r}$, write

$$\delta_s = \max \hat{\gamma}_0 v_s + \sum_{r=1}^{n-s} \hat{\gamma}_r [v_{r+s} - \tfrac{r}{n-s} v_{n-s}]$$

$$\text{s. t. } \sum_{r=0}^{n-s} \hat{\gamma}_r = 1; \quad \hat{\gamma}_r \geq 0; \quad r = 0, 1, \ldots, n\text{-}s$$

Finally,

$$\delta_s = \max \sum_{r=0}^{n-s} \hat{\gamma}_r\left[v_{r+s} - \tfrac{r}{n-s} v_{n-s}\right]$$

$$\text{s.t. } \sum_{r=0}^{n-s} \hat{\gamma}_r = 1; \quad \hat{\gamma}_r \geq 0; \; r = 0, 1, \ldots, n-s$$

$$\text{or } \; \delta_s = \max \left\{v_{r+s} - \tfrac{r}{n-s} v_{n-s} : r = 0, 1, \ldots, n-s\right\}. \quad \Diamond$$

Example 3

If the game in Example 2 is symmetric, then the δ_s are:

$$\delta_1 = \max \begin{Bmatrix} v_2 - \frac{1}{3}v_3; \\ \frac{1}{3}v_3; \\ v_4 - v_3 \end{Bmatrix}; \quad \delta_2 = \max \begin{Bmatrix} v_3 - \frac{1}{2}v_2; \\ v_4 - v_2 \end{Bmatrix}; \quad \delta_3 = v_4 - v_1.$$

III. Classes of Reasonable Outcomes

Milnor's Class D

Using the concept of reasonable demands, Milnor [1952] defines the **class D** of reasonable allocations for any game (N, v), which we denote D(v), as follows:

$$D(v) = \{x \in E: \ x(S) \le \delta(S) \ \text{for all } S \subseteq N\}.$$

The question arises as to whether D(v) is always nonempty. Milnor showed that, if the game is symmetric then D(v) contains the symmetrical outcome, which assigns each player the payoff $x(i) = \frac{1}{n}v(N)$. A proof of this will also be seen in the next section (theorem 3), though in a more general setting. In general, however, the set D can be empty:

Example 4

Shapley [1971] constructed a game with $N = \{P_1, P_2, \ldots, P_{21}\}$ where there are seven "distinguished" coalitions, $R_1, R_2, \ldots, R_7$. The game has the additional property that each player belongs to exactly five of the seven different coalitions, and no two players belong to the same five. So the incidence matrix is of the form:

	P_1	P_2	.....	P_{21}
R_1	1	1	.....	0
R_2	1	1	.....	0
	1	1	.....	1
:	1	1	.....	1
:	1	0	.....	1
	0	1	.....	1
R_7	0	0	.....	1

Note that there are fifteen players in each of the rows. The characteristic function of the game, v, is defined so that

$$\begin{cases} v(\varnothing) = 0; \\ v(S) = |S|-1 & \text{if } \varnothing \subset S \subseteq R_k \text{ for some k} \\ v(S) = |S|-2 & \text{if } S \not\subset R_k \text{ for all k and } S \neq N \\ v(N) = 18 \end{cases}$$

Claim: Shapley's game has $D(v) = \varnothing$. [3]

Proof:

First, we will show that $d(S) = \frac{36}{7}$ is unreasonable for $S = N \backslash R_k$. Notice that $S \not\subset R_k$ for any k, and $|S| = 6$. If y is such that $y(j) = \frac{14}{15}$ for each $j \in R_k$, and $y(S) = 4$, then:

[i] $y(N \backslash S) = y(R_k) = 15\left(\frac{14}{15}\right) = 14 = 15 - 1 = v(R_k)$, and

[ii] for each $T \subseteq R_k$ with $|T| = t$ and $0 \leq t \leq 15$, we have

$$y(T) - v(S \cup T) + d(S) = \tfrac{14}{15}t - (6 + t - 2) + \tfrac{36}{7} = \tfrac{8}{7} - \left(\tfrac{1}{15}\right)t > 0.$$

Thus, by definition, $\frac{36}{7}$ is unreasonable for S because R_k can enforce the allocation y so that none of its members are induced to join with S.

Now suppose $\hat{x} \in D(v)$. Then, $\hat{x}(N \setminus R_k) < \frac{36}{7}$ for all $k = 1, 2, \ldots, 7$. So we have, $36 = 2\hat{x}(N) = \sum_{k=1}^{7} \hat{x}(N \setminus R_k) < 7\left(\frac{36}{7}\right) = 36$, a contradiction. Hence $D(v) = \varnothing$. ◊

The following result, which is a consequence of Proposition 4, gives us a relationship between the sets D(v) and C(v).

Theorem 2: If (N, v) is a CFF game with nonempty core, C(v), then C(v) = D(v).

Proof :

$$\begin{aligned} x \in C(v) &\Leftrightarrow x(S) \geq v(S), \quad \forall S \subseteq N \\ &\Leftrightarrow x(N) - x(S) \leq v(N) - v(S), \quad \forall S \subseteq N \\ &\Leftrightarrow x(N \setminus S) \leq v(N) - v(S), \quad \forall S \subseteq N \\ &\Leftrightarrow x(S) \leq v(N) - v(N \setminus S), \quad \forall S \subseteq N \\ &\Leftrightarrow x(S) \leq \delta(S), \quad \forall S \subseteq N \\ &\Leftrightarrow x \in D(v) \quad ◊ \end{aligned}$$

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3 This result is due to David Housman.

Because we wish to say something about the class **D** of a game whenever its core is empty, we seem to have the motivation to study a more generalized concept of the core. We now present the notion of *strong ε-cores* which Shapley and Shubik introduced (1963), (1966). A strong ε-core can be interpreted as the set of all pre-imputations that cannot be improved upon by any coalition if one imposes a "cost "of ε (or a "bonus" if ε is negative). This ε is also referred to as a *side payment.*

Definition: Let ε be a real number. The **strong ε-core** of the game (N, v), denoted $C_\varepsilon(v)$, is given by:

$$C_\varepsilon(v)=\{x\in E:\ x(S)\geq v(S)-\varepsilon,\quad \forall S\subseteq N\}.$$

Notice that $C_0(v)=C(v)$. Also $C_\varepsilon(v)\supset C_{\varepsilon'}(v)$ whenever $\varepsilon>\varepsilon'$.

One can also define $e(S, x) = v(S) - x(S)$, the **excess** of coalition S with respect to **x**, or more intuitively, the "complaint" of S given the allocation **x**. Then, the ε-core becomes:

$$C_\varepsilon(v)=\{x\in E:\ e(S,x)\leq\varepsilon,\quad \forall S\subseteq N\},$$

the set of all pre-imputations that give rise to complaints no greater than ε.

It should be clear that $C_\varepsilon(v)\neq\varnothing$ for sufficiently large ε, and $C_\varepsilon(v)=\varnothing$ for sufficiently small ε. We now proceed to define some critical values of ε whose cores will be related to class **D**.

Definition: The **least-core** of the game (N, v), to be denoted **LC(v)**, is the intersection of all nonempty strong ε-cores. Equivalently, if we define the **least-ε** to be the smallest ε such that $C_\varepsilon(v)\neq\varnothing$ and denote it ε_0, then $C_{\varepsilon_0}(v)=LC(v)$. Formally, the least-ε is given by $\varepsilon_0=\min\{\ \max\{e(S,x):\ \varnothing\neq S\subset N\}:\ x\in E\ \}$. Clearly, the core is empty if and only if $\varepsilon_0>0$.

Definition: Let $\varepsilon(S)=\delta(S)+v(N\setminus S)-v(N)$, and define $\bar{\varepsilon}=\max\{\varepsilon(S):\ S\subseteq N\}$.

Notice that if the core is nonempty, then $\delta(S) = v(N)-v(N\setminus S)$ and so $\varepsilon(S)=0$ for all S. If, on the other hand, the core is empty, then $\delta(S)\geq v(N)-v(N\setminus S)$ for all $S\subseteq N$, and so $\bar{\varepsilon}\geq 0$.

Proposition 7: For any game (N, v), if $\bar{\varepsilon}<\varepsilon_0$ then **D(v)** is empty.

Proof:

Suppose that $\bar{\varepsilon} < \varepsilon_0$ but $D(v)$ is nonempty. Then, for each $x \in D(v)$ and each $S \subseteq N$, we have:

$$\begin{aligned} & x(S) \le \delta(S) = \varepsilon(S) - v(N \setminus S) + v(N) \\ \Rightarrow\ & x(N \setminus S) \ge v(N \setminus S) - \varepsilon(S) \\ \Rightarrow\ & v(N \setminus S) - x(N \setminus S) \le \varepsilon(S) \le \bar{\varepsilon} < \varepsilon_0 \end{aligned}$$

But this gives $e(S, x) < \varepsilon_0$ for all $S \subseteq N$, which contradicts the minimality of ε_0. So $D(v)$ must be empty. ◊

Proposition 8: If (N, v) is any game, then $D(v) \subseteq C_{\bar{\varepsilon}}(v)$.

Proof:

We may assume $D(v)$ is nonempty and let $x \in D(v)$. Then, for each $S \subseteq N$, we have $x(S) \le \delta(S) = \varepsilon(S) - v(N \setminus S) + v(N)$, which implies $x(N \setminus S) \ge v(N \setminus S) - \varepsilon(S)$. That is, $e(N \setminus S, x) \le \varepsilon(S) \le \bar{\varepsilon}$, for each $S \subseteq N$, thus $x \in C_{\bar{\varepsilon}}(v)$. ◊

Example 5

Consider a game (N, v) with N = {1, 2, 3} and suppose C(v) is empty. Then, we know $\delta(i) = \frac{1}{2}v(ij) + \frac{1}{2}v(ik) - \frac{1}{2}v(jk) > v(ijk) - v(jk)$, and $\delta(jk) = v(ijk) - v(i)$ for each $i \in N$. This gives $\varepsilon(i) = \frac{1}{2}v(ij) + \frac{1}{2}v(ik) + \frac{1}{2}v(jk) - v(ijk) > 0$, and $\varepsilon(N \setminus i) = 0$ for all $i \in N$. So we have $\bar{\varepsilon} = \frac{1}{2}v(ij) + \frac{1}{2}v(ik) + \frac{1}{2}v(jk) - v(ijk)$.

Claim: $D(v)$ is nonempty and $C_{\bar{\varepsilon}}(v) = D(v)$.

Proof:

First, we prove $D(v)$ is nonempty. This can be done by showing $\delta(i) + \delta(j) \le \delta(ij)$ for any pair of players, and $\sum_{i \in N} \delta(i) \ge v(ijk)$. Since the core is empty, we know $\delta(i) = \frac{1}{2}[v(ij) + v(ik) - v(jk)]$, and $\delta(j) = \frac{1}{2}[v(ij) + v(jk) - v(ik)]$. Then, $\delta(i) + \delta(j)$ = $v(ij) \le v(ijk) - v(k) = \delta(ij)$. This last inequality follows by superadditivity.

Now we verify:

$$\begin{aligned}\textstyle\sum_{i\in N}\delta(i) = {}&\tfrac{1}{2}[v(ij)+v(ik)-v(jk)]+\\ &\tfrac{1}{2}[v(ij)+v(jk)-v(ik)]+\\ &\tfrac{1}{2}[v(ik)+v(jk)-v(ij)]=\\ ={}&\tfrac{1}{2}[v(ij)+v(ik)+v(jk)]>v(ijk)\end{aligned}$$

The last inequality follows from the assumption that the core is empty. So, if we define $\mathbf{x}$ so that $\mathbf{x}(i)=\delta(i)-\frac{1}{3}[\sum_{i\in N}\delta(i)-v(ijk)]$, for each $i\in N$, we have $\mathbf{x}\in \mathbf{D}(v)$.

Since $\mathbf{D}(v)$ is not empty, Proposition 7 implies $\bar{\varepsilon}\geq\varepsilon_0$ and Proposition 8 gives $\mathbf{C}_{\bar{\varepsilon}}(v)\supseteq \mathbf{D}(v)$, so we need only show $\mathbf{C}_{\bar{\varepsilon}}(v)\subseteq\mathbf{D}(v)$. Let $\mathbf{x}\in\mathbf{C}_{\bar{\varepsilon}}(v)$. It follows that, for each $i\in N$,

$$\left.\begin{matrix}e(i,\mathbf{x})\leq\bar{\varepsilon}\\ e(jk,\mathbf{x})\leq\bar{\varepsilon}\end{matrix}\right\}\Leftrightarrow\left.\begin{matrix}\mathbf{x}(i)\geq v(i)-\bar{\varepsilon}\\ \mathbf{x}(jk)\geq v(jk)-\bar{\varepsilon}\end{matrix}\right\}\Leftrightarrow\begin{cases}(1)\ \mathbf{x}(jk)\leq v(ijk)-v(i)+\bar{\varepsilon}\\ (2)\ \mathbf{x}(i)\leq v(ijk)-v(jk)+\bar{\varepsilon}\end{cases}$$

So,

$$\begin{aligned}(2)\ \Leftrightarrow\ \mathbf{x}(i)&\leq v(ijk)-v(jk)+\tfrac{1}{2}v(ij)+\tfrac{1}{2}v(ik)+\tfrac{1}{2}v(jk)-v(ijk)\\ &=\tfrac{1}{2}v(ij)+\tfrac{1}{2}v(ik)-\tfrac{1}{2}v(jk)=\delta(i)\end{aligned}$$

Also,

$$\begin{aligned}(2)\ \Rightarrow\ \mathbf{x}(jk)&\leq 2v(ijk)-v(ik)-v(ij)+[v(ij)+v(ik)+v(jk)-2v(ijk)]\\ &=v(jk)\leq v(ijk)-v(i)=\delta(jk)\end{aligned}$$

Thus, $\mathbf{x}(S)\leq\delta(S)$ for all $S\subseteq N=\{1,2,3\}$, and so $\mathbf{x}\in\mathbf{D}(v)$. ◊

Example 6

Consider a symmetric game with $N = \{1, 2, 3, 4\}$, and empty core. From Example 3, we know:

$$\delta_1=\max\begin{Bmatrix}v_2-\frac{1}{3}v_3;\\ \frac{1}{3}v_3;\\ v_4-v_3\end{Bmatrix};\quad \delta_2=\max\begin{Bmatrix}v_3-\frac{1}{2}v_2;\\ v_4-v_2\end{Bmatrix};\quad \delta_3=v_4-v_1.$$

Since the core is empty, we also know $nv_s>sv_n$, for some s. Observe, however, that we could not have $4v_1>v_4$, nor $4v_2>2v_4$, since superadditivity is violated in either case. So we must have $4v_3>3v_4$, which implies $\frac{1}{3}v_3>\frac{1}{4}v_4\geq\frac{1}{2}v_2$. Using these inequalities, it can be shown easily that $\delta_1=\frac{1}{3}v_3$.

Claim: D(v) is uniquely determined by δ_1; that is, $2\delta_1\leq\delta_2$ and $3\delta_1\leq\delta_3$.

Proof:

By superadditivity, $3\delta_1 = v_3 \le v_4 - v_1 = \delta_3$. If $\delta_2 = v_3 - \frac{1}{2}v_2$, then $2\delta_1 = \frac{2}{3}v_3 \le v_3 - \frac{1}{2}v_2$ if and only if $\frac{1}{2}v_2 \le \frac{1}{3}v_3$ which holds true by assumption. Now, suppose $\delta_2 = v_4 - v_2$ so that $v_4 - v_2 \ge v_3 - \frac{1}{2}v_2$. Then, we have

$$v_4 \ge \tfrac{1}{2}v_2 + v_3 = \tfrac{1}{2}v_2 + \tfrac{3}{3}v_3 \ge \tfrac{1}{2}v_2 + \tfrac{1}{2}v_2 + \tfrac{2}{3}v_3 = v_2 + \tfrac{2}{3}v_3.$$

Thus, $\frac{2}{3}v_3 \le v_4 - v_2$ and so $2\delta_1 \le \delta_2$. ◊

Notation: In a symmetric game, we use ε_s to denote $\varepsilon(S)$ for $S \subseteq N$ with $|S| = s$. That is, $\varepsilon_s = \delta_s + v_{n-s} - v_n$.

Claim: $D(v) = C_{\bar{\varepsilon}}(v) = C_{\varepsilon_1}(v)$.

Proof:

Simple calculations show $\varepsilon_1 = \frac{4}{3}v_3 - v_4 > 0$, $\varepsilon_2 = \max\{v_3 + \frac{1}{2}v_2 - v_4,\ 0\}$, and $\varepsilon_3 = 0$. But, $\varepsilon_1 = \frac{4}{3}v_3 - v_4 \ge \frac{3}{3}v_3 + \frac{1}{2}v_2 - v_4 = \varepsilon_2$, thus $\bar{\varepsilon} = \varepsilon_1$.

Proposition 6 gives $D(v) \subseteq C_{\bar{\varepsilon}}(v)$, so it remains to show $C_{\bar{\varepsilon}}(v) \subseteq D(v)$.

Suppose $x \in C_{\bar{\varepsilon}}(v)$. By the previous claim, we need only prove $x(i) \le \delta_1$. Because $x(N \setminus S) \ge v_{n-s} - \bar{\varepsilon}$ for any value of s, and $s = |S|$, then $x(S) \le v_n - v_{n-s} + \bar{\varepsilon}$. Therefore, $x(i) \le v_4 - v_3 + \bar{\varepsilon} = v_4 - v_3 + \frac{4}{3}v_3 - v_4 = \frac{1}{3}v_3 = \delta_1$, and so $C_{\bar{\varepsilon}}(v) \subseteq D(v)$. ◊

In general, for a given game (N, v) we cannot always find an ε for which $C_\varepsilon(v) = D(v)$. We illustrate this with the next example.

Example 7:

Consider the CFF game where N={i, j, k, l} and characteristic function v defined as follows:

$v(i) = 0$	$v(ij) = 15$	$v(ijk) = 30$
$v(j) = 0$	$v(ik) = 0$	$v(ijl) = 30$
$v(k) = 0$	$v(il) = 15$	$v(ikl) = 15$
$v(l) = 0$	$v(jk) = 15$	$v(jkl) = 15$
	$v(jl) = 0$	
	$v(kl) = 15$	$v(ijkl) = 30$

Solving the linear program:

$$\begin{aligned} \varepsilon_0 = & \min \varepsilon \\ \text{s.t.} \quad & e(S,x) \le \varepsilon \quad \forall S \subseteq N \\ & x \in \mathrm{IR}(v) \end{aligned}$$

we obtain $\varepsilon_0 = 5$. Also, using Example 2 to compute each $\delta(S)$ gives:

$\delta(i) = 15$	$\delta(ij) = 22\frac{1}{2}$	$\delta(ijk) = 30$
$\delta(j) = 15$	$\delta(ik) = 30$	$\delta(ijl) = 30$
$\delta(k) = 7\frac{1}{2}$	$\delta(il) = 15$	$\delta(ikl) = 30$
$\delta(l) = 7\frac{1}{2}$	$\delta(jk) = 15$	$\delta(jkl) = 30$
	$\delta(jl) = 30$	
	$\delta(kl) = 15$	

Now, notice that an allocation **x** is in $C_5(v)$ if and only if, for each $S \subseteq N$, $v(S) - 5 \le x(S) \le v(N) - v(N \setminus S) + 5$. And if this **x** is to be in $D(v)$ as well, it must satisfy $v(N) - \delta(N \setminus S) \le x(S) \le \delta(S)$ for all $S \subseteq N$. Then, it can be checked easily that $x = (11, 9, 5, 5) \in C_5(v)$. However, $x = (11, 9, 5, 5) \notin D(v)$ since we do not have $15 \le x(il) \le 15$.

$D(v)$ is not empty since $(10, 10, 5, 5) \in D(v)$. So if ε is such that $C_\varepsilon(v) = D(v)$, then $\varepsilon \ge 5$. But this gives that $C_5(v) \subseteq C_\varepsilon(v) \subseteq D(v)$ which is not satisfied for our **x** above. Thus, no such ε exists.

Other Classes of Reasonable Outcomes

One could argue that the symmetrical allocation is at least intuitively "reasonable" for Shapley's game in Example 4, although we saw that no coalition S of size six could possibly receive a reasonable payoff. The problem was that the complement could always enforce some allocation which gave S less than what they demanded, if this was more than v(S). It would appear, then, that no coalitions of size six would form during negotiations, since it would not be profitable for its members. Based on the idea that the formation of certain coalitions may not be profitable, we consider other classes of reasonable outcomes.

Definition: For any game (N, v), let $D_s(v) = \{x \in E: \ x(S) \le \delta(S), \ S \subseteq N, \ |S| = s\}$.

Notice that $\mathbf{D}(v)$ is the intersection of all $\mathbf{D}_s(v)$. In general, we do not know when each $\mathbf{D}_s(v)$ is empty, but we find that at least two of them are always nonempty.

Proposition 9: If (N, v) is any game with $n > 1$, then $\mathbf{D}_{n-1}(v) = \mathbf{IR}(v) \neq \varnothing$.

Proof:

By Proposition 2, we know that $\delta(N \setminus i) = v(N) - v(i)$, for any $i \in N$. So if $\mathbf{x} \in \mathbf{E}$ is any allocation satisfying $\mathbf{x}(N \setminus i) \leq \delta(N \setminus i) = v(N) - v(i)$, then it follows that $\mathbf{x}(i) \geq v(i)$, and we have that $\mathbf{x}$ is individually rational. The fact that $\mathbf{IR}(v)$ is nonempty follows from superadditivity. ◊

Proposition 10: If (N, v) is any game, then $\mathbf{D}_1(v) \neq \varnothing$.

Proof:

It suffices to show that $\sum_{i \in N} \delta(i) \geq v(N)$, since this gives that the allocation $\mathbf{x}$ defined by $\mathbf{x}(i) = \delta(i) - \frac{1}{n}\left(\sum_{i \in N} \delta(i) - v(N)\right)$ is contained in $\mathbf{D}(v)$.

Let us define $\lambda_0 = \frac{n-1}{n}$, $\lambda_R = \frac{1}{n}$ for $R = N \setminus i$ and for $R = N \setminus \{i, j\}$, where $j \neq i$, and $\lambda_R = 0$ for all other $R \subset N \setminus S$. This is a feasible solution for LP2 since, for $j \neq i$, $\sum_{R \ni j} \lambda_R = \frac{1}{n} + \frac{n-2}{n} = \frac{n-1}{n} = \lambda_0$, and $\sum_{R \subseteq N \setminus i} \lambda_R = \frac{1}{n} + \frac{n-1}{n} = 1$. Then, $\delta(i) \geq -\frac{n-1}{n} v(N \setminus i) + \frac{1}{n} v(N) + \frac{1}{n} \sum_{j \neq i} v(N \setminus j) = -v(N \setminus i) + \frac{1}{n} v(N) + \frac{1}{n} \sum_{j \in N} v(N \setminus j)$, and so $\sum_{i \in N} \delta(i) \geq -\sum_{i \in N} v(N \setminus i) + v(N) + \sum_{j \in N} v(N \setminus j) = v(N)$. ◊

IV. Reasonableness and Games in Partition Function Form

Games in Partition Function Form (PFF) differ from those in Coalition Function Form (CFF) in that the worth assigned to each coalition depends on how the rest of the players collude. We begin with the set of players $N = \{1, 2, \ldots, n\}$ and introduce the following concepts:

Definition: A collection $P = \{S_1, S_2, \ldots, S_m\}$ is a **partition** of N if and only if the following hold: (1) $\varnothing \neq S_k \subseteq N$ for each $k = 1,\ldots,m$; (2) for each $i \in N$, there exists k so that $i \in S_k$; (3) $S_k \cap S_j = \varnothing$ for all $k \neq j$.

Notation: The set of all partitions of N is denoted **PT**, and the set of all partitions *containing* coalition S, where $S \subseteq N$, is denoted **PT(S)**.

Definition: An **embedded coalition** is a pair (S;P), where $S \subseteq N$, and $P \in PT(S)$. The set of all embedded coalitions will be denoted **ECL.**

Definition: A PFF game is a pair (N, w) with $w \in \mathbf{R}^{ECL}$, where w(S;P) represents the amount, or worth, that coalition S would receive if partition P formed.

Definition: A PFF game (N, w) is **superadditive** if, for any fixed partition $P \in PT$ and sets $S, T \in P$, we have $w(S;P) + w(T;P) \leq w(S \cup T;Q)$, where Q is the partition obtained from P by joining the sets S and T.

All the games treated in this section will be superadditive, PFF games, unless stated otherwise.

Definition: A partition Q is a **refinement** of partition P if, for each $R \in Q$ there exists $S \in P$ such that $R \subseteq S$.

Definition: A PFF game (N, w) is **partition-monotonic** if $w(S;P) \leq w(S;Q)$ whenever $P, Q \in PT(S)$ and Q is a refinement of P.

Definition: A PFF game (N, w) is **symmetric** if permuting the players in any ECL does not change the worth function.

Definition: Suppose $S \subseteq N$ is given. Let P^S denote the partition in $\mathbf{PT}(S)$ such that $w(S;P^S) = \max\{w(S;P)\colon \forall P \in \mathbf{PT}(S)\}$, and let P_S denote the partition in $\mathbf{PT}(S)$ such that $w(S;P_S) = \min\{w(S;P)\colon \forall P \in \mathbf{PT}(S)\}$. Notice that if (N, w) is partition-monotonic, then P^S is of the form $\{S, \{i_1\}, \{i_2\}, \ldots, \{i_{n-s}\}\}$, where $i_k \in N \setminus S$ for each k = 1, 2,..., n-s, and P_S is of the form {S, N\S}. We are using |N| = n, and |S| = s.

Example 8

The following is a symmetric, PFF game with N = {1, 2, 3}

S	P	w(S;P)
123	123	9
ij	ij, k	5
i	i, j, k	0
i	i, jk	4

Notice that this game is not partition-monotonic.

Allocations are defined in the same way for PFF games and CFF games, and again, we are only interested in those outcomes that satisfy efficiency. The set all of pre-imputations for PFF games will be defined $\overline{\mathbf{E}} = \{x \in \mathbf{R}^n\colon x(N) = w(N;N)\}$. We now extend Milnor's concept of reasonable demands to games in partition function form.

Definition: Let d(S; P) denote the payoff that coalition $S \in P$ demands in a PFF game (N, w) with $(S;P) \in \mathbf{ECL}$. Then d(S; P) is **unreasonable** for $S \in P$ if:

[a] $w(S;P) < d(S;P)$, and

[b] there exists an allocation $x \in \overline{\mathbf{E}}$ such that:

[i] $x(R) \leq w(R;P)$ for all $S \neq R \in P$, and

[ii] $x(T) > \max\{w(S \cup T;Q) - d(S;P)\colon Q \in \mathbf{PT}(S)\}$ for any nonempty $T \subseteq N \setminus S$.

Condition [a] says that S is demanding more than its corresponding worth in P. Condition [b] says that this partition is a "stable" one, in the sense that there are no incentives for any subset of N\S to join with S, knowing S will keep d(S; P). So the allocation that N\S is enforcing on S is feasible (condition [i]), and it is also such that S will not be able to disrupt the partition by causing defections to occur (condition [ii]).

Definition: A demand d(S) is **unreasonable** for S if there exists a partition $P \in PT(S)$ such that d(S) is unreasonable for $S \in P$.

Definition: A payoff for coalition S is **reasonable** if it is not unreasonable.

Given a PFF game, we wish to determine the **maximum reasonable payoff**, $\delta(S)$, that a coalition S can demand in the game. Since all partitions $P \in PT(S)$ must be considered, we have that $\delta(S) = \min\{\delta(S;P) : P \in PT(S)\}$, where $\delta(S;P)$ is the maximum reasonable payoff that S can demand in P. In general, $\delta(S;P)$ can be found by solving the following linear program:

$$\begin{aligned} \delta(S;P) &= \min\ \delta \\ \text{s.t.}\quad & \delta \geq w(S;P) \\ & -x(R) \geq -w(R;P) \qquad S \neq R \in P \\ & \delta + x(T) \geq w(S \cup T;\ Q^{S \cup T}) \qquad \emptyset \neq T \subseteq N \setminus S \end{aligned} \tag{LP3}$$

Again, the dual of this program will be of greater use in this paper, so we write:

$$\begin{aligned} \delta(S;P) &= \max\ \lambda_0 w(S;P) - \sum_{S \neq R \in P} \gamma_R w(R;P) + \sum_{T \subseteq N \setminus S} \lambda_T w(S \cup T;\ Q^{S \cup T}) \\ \text{s.t}\quad & \sum_{\emptyset \neq T \subseteq N \setminus S} \lambda_T + \lambda_0 = 1 \\ & \sum_{\substack{R \ni i \\ S \neq R \in P}} \gamma_R = \sum_{\substack{T \ni i \\ T \subseteq N \setminus S}} \lambda_T \quad \text{for } i \in N \setminus S \\ & \lambda_0,\ \gamma_R,\ \lambda_T \geq 0 \end{aligned} \tag{LP4}$$

Proposition 11: If (N, w) is any PFF game, then $\delta(N \setminus i) = w(N;N) - w(i;\ \{i, N \setminus i\})$ for any $i \in N$.

Proof:

Since there is only one partition to consider, namely P = $\{N\setminus i,\ i\}$, we know $\delta(N \setminus i) = \delta(N \setminus i;\ \{i, N \setminus i\})$. Now, suppose that δ is unreasonable for (N\i; P). Then,

[a] $w(N\setminus i;\ P) < \delta$, and

[b] there exists $x \in \overline{E}$ such that

  [i] $x(i) \leq w(i;\ P)$, and

  [ii] $x(i) > w(N;\ N) - \delta$

These conditions hold if and only if

(a) $w(N\backslash i; P) < \delta$, and (b) $w(i; P) > w(N; N) - \delta \Leftrightarrow \delta > w(N; N) - w(i; P)$.

However, (a) is redundant since superadditivity gives $w(N; N) - w(i; P) \geq w(N\backslash i; P)$.

Thus, $\delta(i) = w(N; N) - w(i; P)$. ◊

Definition: The set of all reasonable outcomes for a given PFF game (N, w) is:

$$\overline{D}(w) = \{x \in \overline{E}: \ x(S) \text{ is reasonable, } S \subseteq N\}.$$

Proposition 12: Suppose (N, v) is a CFF game. Define the PFF game (N, w) by setting $w(S;P) = v(S)$ for each (S;P). Then the following hold:

(1) (N, w) is partition-monotonic;

(2) If (N, v) is superadditive, then (N, w) is also superadditive.

(3) $D(v) = \overline{D}(w)$

Proof:

(1) This follows directly from the definition of partition-monotonicity, since $w(S; P) = w(S; Q)$ for all $P, Q \in PT(S)$.

(2) Let $S, S' \subseteq N$ be two disjoint sets. Then,

$$v(S \cup S') \geq v(S) + v(S') \Leftrightarrow w(S \cup S';Q) \geq w(S;P) + w(S';P)$$

for any $P \in PT(S, S')$ and any $Q \in PT(S \cup S')$.

(3) We have that $y \in D(v)$ if and only if $y(S)$ is reasonable for all $S \subseteq N$; that is, for each $S \subseteq N$ and each $x \in E$, one of the following occurs:

[i] $x(N\backslash S) > v(N\backslash S)$, or [ii] $x(T) \leq v(S \cup T) - y(S)$ for some $T \subseteq N \setminus S$.

But [i] $\Leftrightarrow x(N\backslash S) > w(N\backslash S;P)$ for any $P \in PT(N \setminus S)$, and

[ii] $\Leftrightarrow x(T) \leq w(S \cup T;Q) - y(S)$ for all $Q \in PT(S \cup T)$

Therefore, $y(S)$ is reasonable for each $(S;P) \in ECL$, and so $y \in D(v)$ if and only if $y \in \overline{D}(w)$. ◊

Clearly, in view of Shapley's game and the previous proposition, $\overline{D}(w)$ is sometimes empty. In fact, this set could be empty for much simpler games.

Example 9

Reconsider the game presented in Example 8:

S	P	w(S;P)
123	123	9
ij	ij, k	5
i	i, j, k	0
i	i, jk	4

By Proposition 11, we have $\delta(ij) = 9 - 4 = 5$ for any pair of players. To compute $\delta(i)$, we first let P = {i, jk} and use LP4 to write:

$$\begin{aligned} \delta(i;P) = \max\ & 4\lambda_0 - 5\gamma_{jk} + 5\lambda_j + 5\lambda_k + 9\lambda_{jk} \\ \text{s.t.}\ & \lambda_0 + \lambda_j + \lambda_k + \lambda_{jk} = 1 \\ & \gamma_{jk} - \lambda_j \qquad - \lambda_{jk} = 0 \\ & \gamma_{jk} \qquad - \lambda_k - \lambda_{jk} = 0 \\ & \lambda_0, \gamma_{jk}, \lambda_j, \lambda_k, \lambda_{jk} \geq 0 \end{aligned}$$

which reduces to:

$$\begin{aligned} \delta(i;P) &= \max\ 4\lambda_0 - 5(\lambda_j + \lambda_{jk}) + 5(\lambda_j + \lambda_k) + 9\lambda_{jk} \\ &= \max\ 4(\lambda_0 + \lambda_k + \lambda_{jk}) + \lambda_k \\ \text{s.t.}\ & \lambda_0 + \lambda_j + \lambda_k + \lambda_{jk} = 1 \\ & +\lambda_j \qquad + \lambda_{jk} = \gamma_{jk} \\ & \qquad +\lambda_k + \lambda_{jk} = \gamma_{jk} \\ & \lambda_0, \gamma_{jk}, \lambda_j, \lambda_k, \lambda_{jk} \geq 0 \end{aligned}$$

and so $\delta(i;P) = 4$.

If P = {i, j, k} then we obtain from LP4:

$$\begin{aligned} \delta(i;P) = \max\ & 0\lambda_0 - 0\gamma_j - 0\gamma_k + 5\lambda_j + 5\lambda_k + 9\lambda_{jk} \\ \text{s.t.}\ & \lambda_0 + \lambda_j + \lambda_k + \lambda_{jk} = 1 \\ & \gamma_j - \lambda_j \qquad - \lambda_{jk} = 0 \\ & \gamma_k \qquad - \lambda_k - \lambda_{jk} = 0 \\ & \lambda_0, \gamma_j, \gamma_k, \lambda_j, \lambda_k, \lambda_{jk} \geq 0 \end{aligned}$$

which reduces to:

$$\begin{aligned}\delta(i;P) = \max \quad & 5\lambda_j + 5\lambda_k + 9\lambda_{jk} \\ = \max \quad & 5(1-\lambda_0) + 4\lambda_{jk} \\ \text{s.t.} \quad & \lambda_0 + \lambda_j + \lambda_k + \lambda_{jk} = 1 \\ & \lambda_j \qquad + \lambda_{jk} = \gamma_j \\ & \qquad \lambda_k + \lambda_{jk} = \gamma_k \\ & \lambda_0, \gamma_j, \gamma_k, \lambda_j, \lambda_k, \lambda_{jk} \geq 0\end{aligned}$$

and so $\delta(i;P) = 9$. Thus, we now get $\delta(i) = 4$.

Now, since any $x \in \overline{D}(w)$ satisfies $w(N;N) - \delta(N \setminus S) \leq x(S) \leq \delta(S)$, for each $S \subseteq N$, then we must have $4 \leq x(i) \leq 4$ for each player i, and $5 \leq x(ij) \leq 5$ for each pair. These inequalities are clearly inconsistent, and so $\overline{D}(w)$ must be empty.

The following result says that, whenever a PFF game is partition-monotonic, we need only consider a unique partition in PT(S) to determine whether d(S) is reasonable for $S \subseteq N$.

Proposition 13: Suppose (N, w) is a partition-monotonic game and let $S \subseteq N$ be given, so that $P_S = \{S, N \setminus S\}$. If $d(S;P_S)$ is reasonable for $S \in P_S$, then it is reasonable for (S; P), where P is any partition in PT(S). That is, $\delta(S) = \delta(S;P_S)$.

<u>Proof</u>:

If $d(S;P_S)$ is reasonable for $S \in P_S$, then one of the following occurs:

[a] $d(S;P_S) \leq w(S;P_S)$, or

[b] for each $x \in \overline{E}$, we have

[i] $x(N\setminus S) > w(N\setminus S; P_S)$ or

[ii] $x(T) \leq w(S \cup T;Q) - d(S;P_S)$ for some $T \subseteq N \setminus S$, and some $Q \in PT(S \cup T)$.

If [a] is the case, then $d(S;P_S) \leq w(S;P_S) \leq w(S;P)$ for all $P \in PT(S)$, and so $d(S;P_S)$ is reasonable with respect to all $P \in PT(S)$.

Suppose [b] is the case. If [ii] holds, then we are done since this relationship is independent of partition P. If [i] holds, then superadditivity implies

$$x(N \setminus S) > w(N \setminus S;P_S) \geq \sum_{k=1}^{m} w(R_k;P),$$

where P is any refinement of P_S of the form $P = \{R_1, R_2, \ldots, R_m, S\}$.

Then, there exists some $R_j \in P$ such that $x(R_j) > w(R_j;P)$. That is, for each $P \in \mathbf{PT}(S)$, and each $x \in \overline{\mathbf{E}}$, some payoff $x(R)$ is unfeasible for $R \in P$, so, again, $d(S;P_S)$ is reasonable for S with respect to all $P \in \mathbf{PT}(S)$. ◊

A similar conclusion cannot always be made for games that are not partition-monotonic. This can be seen in Example 9 where $\delta(i;P_i) = \delta(i;\{j,k\}) = 9$, but $\delta(i;P^i) = 4$, and so $\delta(i) \neq \delta(i;P_i)$.

Proposition 14: Assume (N, w) is a partition-monotonic, PFF game. Define a CFF game (N, v) by setting $v(S) = w(S;P_S)$ for each $S \subseteq N$. Then $\mathbf{D}(v) \subseteq \overline{\mathbf{D}}(w)$.

Proof:

Let $y \in \mathbf{D}(v)$. Since y(S) is reasonable for all $S \subseteq N$, then for each $x \in \mathbf{E}$ one of the following occurs: [i] $x(N \setminus S) > v(N \setminus S)$, or [ii] $x(T) \leq v(S \cup T) - y(S)$ for some $T \subseteq N \setminus S$. Condition [i] says $x(N \setminus S) > w(N \setminus S;P_S)$, which implies y(S) is reasonable for S with respect to P_S. Then, by Proposition 12, y(S) is reasonable with respect to all $P \in \mathbf{PT}(S)$. Condition [ii] says $x(T) \leq w(S \cup T;Q_{S \cup T}) - y(S)$, where $Q_{S \cup T} = \{S \cup T, N \setminus (S \cup T)\}$, and so y(S) is reasonable with respect to $P \in \mathbf{PT}(S)$. Therefore, $y \in \overline{\mathbf{D}}(w)$. ◊

Example 10

We begin with the set of players N = {1, 2, 3, 4}. Define a symmetric PFF game with characteristic function w, and then a CFF game as in Proposition 13 with characteristic function v:

S	P	w(S;P)
1234	1234	12
ijk	ijk, l	10
ij	ij, k, l	8
ij	ij, kl	5
i	i, j, k, l	2
i	i, jk, l	1
i	i, jkl	0

→

S	v(S)
ijkl	12
ijk	10
ij	5
i	0

The $\delta(S)$ are, with respect to $\mathbf{w}$: $\delta(i) = 4\frac{2}{3}$ $\delta(ij) = 7\frac{1}{2}$ $\delta(ijk) = 12$, and with respect to $\mathbf{v}$: $\delta(i) = 3\frac{1}{3}$ $\delta(ij) = 7\frac{1}{2}$ $\delta(ijk) = 12$. The later values can be obtained from Example 3.

Notice that $\mathbf{x} = (3\frac{1}{2}, 3\frac{1}{2}, 2, 2) \in \overline{\mathbf{D}}(\mathbf{w}) \setminus \mathbf{D}(\mathbf{v})$, so that the inclusion above is strict.

Claim: Let (N, **w**) be any partition-monotonic, PFF game with |N| = 3. If we define a CFF game (N, **v**) by setting $\mathbf{v}(S) = \mathbf{w}(S;P_S)$ for each $S \subseteq N$, then $\mathbf{D}(\mathbf{v}) = \overline{\mathbf{D}}(\mathbf{w})$.

Proof:

By the previous proposition, we need only show $\overline{\mathbf{D}}(\mathbf{w}) \subseteq \mathbf{D}(\mathbf{v})$. So let $\mathbf{y} \in \overline{\mathbf{D}}(\mathbf{w})$. Since $\mathbf{y}(S)$ is reasonable for all $S \subseteq N$, we know $\mathbf{y}(S)$ is reasonable for $S \in P_S$ so that, for each $\mathbf{x} \in \overline{\mathbf{E}}$, either one of the following holds:

[i] $\mathbf{x}(N \setminus S) > \mathbf{w}(N \setminus S;P_S)$ or, [ii] $\mathbf{x}(T) \le \mathbf{w}(S \cup T;Q) - \mathbf{y}(S)$ for some $T \subseteq N \setminus S$, and some $Q \in \mathbf{PT}(S \cup T)$.

If [i] holds, then $\mathbf{y}(S)$ is reasonable for $S \subseteq N$ with respect to the CFF game **v** since $\mathbf{x}(N \setminus S) > \mathbf{w}(N \setminus S;P_S) = \mathbf{v}(N \setminus S)$. If [ii] is the case, then notice that, since n = 3, $|S \cup T| = 2$, or $|S \cup T| = 3$. But any pair of players in a three-player, PFF game belong to a unique partition, and the same is true for the grand coalition. Thus, $\mathbf{w}(S \cup T;Q) = \mathbf{v}(S \cup T)$ is always true, and we have that $\mathbf{x}(T) \le \mathbf{v}(S \cup T) - \mathbf{y}(S)$. Again, this gives that $\mathbf{y}(S)$ is reasonable for $S \subseteq N$ with respect to **v**, and so $\mathbf{y} \in \mathbf{D}(\mathbf{v})$. ◊

Theorem 3: If a PFF game (N, **w**) is symmetric and partition-monotonic, then $\overline{\mathbf{D}}(\mathbf{w}) \neq \emptyset$.

Proof:

We want to show $\frac{s}{n}\mathbf{w}(N;N)$ is reasonable for all $S \subseteq N$. If not, then for some $S \subseteq N$ and for $P_S = \{S, N \setminus S\}$, we know there exists an allocation $\mathbf{x} \in \overline{\mathbf{E}}$ such that:

[i] $\mathbf{x}(N \setminus S) \le \mathbf{w}(N \setminus S;P_S)$, and

[ii] $\mathbf{x}(T) > \mathbf{w}(S \cup T;P^{S \cup T}) - \frac{s}{n}\mathbf{w}(N;N)$, for each $T \subseteq N \setminus S$.

We may choose $\mathbf{x} \in \overline{\mathbf{E}}$ satisfying $\mathbf{x}(j) = \frac{1}{n-s}\mathbf{w}(N \setminus S;P_S)$ for each $j \in N \setminus S$, so that we have equality in [i]. This is because, if T and T* are subsets of N\S with |T| = |T*|, we have $w(S \cup T; P^{S\cup T}) = w(S \cup T^*; P^{S\cup T^*})$ and so [ii] still holds if payoffs are distributed so that $x(T) = x(T^*)$.

Notice that if T=N\S, our choice of x and [ii] imply:

$$(*) \quad w(N \setminus S; P_S) > \tfrac{n-s}{n} w(N;N).$$

Case 1: $n > 2s$

Choose $T \subseteq N \setminus S$ so that $|T| = t = n-2s > 0$. Then $|S \cup T| = n-s$ and, by symmetry, we have $w(S \cup T; P^{S\cup T}) = w(N \setminus S; P^{N\setminus S})$. So [ii] becomes:

$$\tfrac{n-2s}{n-s} w(N \setminus S; P_S) > w(N \setminus S; P^{N\setminus S}) - \tfrac{s}{n} w(N;N).$$

By definition of $P^{N\setminus S}$ we have $w(N \setminus S; P^{N\setminus S}) \geq w(N \setminus S; P_S)$, so

$$\tfrac{n-2s}{n-s} w(N \setminus S; P_S) > w(N \setminus S; P_S) - \tfrac{s}{n} w(N;N)$$

$$\Leftrightarrow \quad \tfrac{s}{n} w(N;N) > \tfrac{s}{n-s} w(N \setminus S; P_S)$$

$$\Leftrightarrow \quad \tfrac{n-s}{n} w(N;N) > w(N \setminus S; P_S).$$

This contradicts (*) above.

Case 2: $n \leq 2s$

If $n - 2s \leq 0$ so that $n - s \leq s < n$, then $n - s < s + t \leq n$ for $t = 1, 2, \ldots, n-s$. There exist nonnegative integers k and r such that $n = k(n-s) + r$ and $0 \leq r < n-s$. So choose $T \subseteq N \setminus S$ with $|T| = t = n-r-s = k(n-s)-s$, or equivalently, $t + s = k(n-s)$. Observe that $1 \leq t \leq n-s$. For such T, then, superadditivity gives

$$w(S \cup T; P^{S\cup T}) \geq k w(N \setminus S; Q)$$

where Q is a partition consisting of k sets of size $(n-s)$, and r singletons, each of which is a member of $N \setminus (S \cup T)$. Notice that Q is a refinement of $P_S = \{S, N \setminus S\}$, and so

$$\tfrac{t}{n-s} w(N \setminus S; Q) \geq \tfrac{t}{n-s} w(N \setminus S; P_S).$$

Now [ii] becomes:

$$\tfrac{t}{n-s} w(N \setminus S; Q) > x(T) = \tfrac{t}{n-s} w(N \setminus S; P_S) > w(S \cup T; P^{S\cup T}) - \tfrac{s}{n} w(N;N)$$

$$\Leftrightarrow \quad \tfrac{t}{n-s} w(N \setminus S; Q) > k w(N \setminus S; Q) - \tfrac{s}{n} w(N;N)$$

$$\Leftrightarrow \quad \tfrac{t}{n-s} w(N \setminus S; Q) > \tfrac{t+s}{n-s} w(N \setminus S; Q) - \tfrac{s}{n} w(N;N)$$

$$\Leftrightarrow \quad \tfrac{s}{n} w(N;N) > \tfrac{s}{n-s} w(N \setminus S; Q)$$

$$\Leftrightarrow \quad \tfrac{n-s}{n} w(N;N) > w(N \setminus S;Q)$$

$$\Rightarrow \quad \tfrac{n-s}{n} w(N;N) > w(N \setminus S;P_s)$$

This contradicts (*) once again, and therefore $\frac{s}{n} w(N;N)$ is reasonable for all $S \subseteq N$. Then, the symmetrical allocation is contained in $\overline{D}(w)$. ◊

Conclusion

As a desirable property for allocations of CFF games, Milnor's notion of reasonableness is, in general, less restrictive than that of group rationality. That is, the core is always a subset of Milnor's class D. We found that if a game is balanced, the core coincides precisely with Milnor's class **D** of reasonable outcomes. This can be seen geometrically, since the core is determined by the hyperplanes $x(S) \geq v(S)$ for $S \subseteq N$; but this is equivalent to $x(N \setminus S) \leq v(N) - v(S) = \delta(N \setminus S)$, which gives the hyperplanes that determine the set **D**.

Whenever a game is not balanced, we are able to find a value of ε, namely $\bar{\varepsilon} = \max\{\delta(S) + v(N \setminus S) - v(N): S \subseteq N\}$, so that $D(v) \subseteq C_{\bar{\varepsilon}}(v)$. Whether the strong $\bar{\varepsilon}$-core is the smallest set with this property is left unresolved. Two special cases are found, however, of games which are not balanced and have class **D** equal to the strong $\bar{\varepsilon}$-core. These are all three-player games, and all four-player symmetric games. In general, it is not known under what circumstances we might have some value of ε for which the strong ε- core is equal to the set **D**.

Milnor's class **D** is sometimes empty, as illustrated with a twenty-one-player game defined by Shapley [1971]. This is the only game we know with **D** being empty, and it would be of interest to determine whether this is the smallest game with such property. While we have shown that **D** is nonempty for all three-player games, we do not know if this remains true for four-player games.

We also define, based on Milnor's concept of reasonable demands, other classes of reasonable outcomes that are less restrictive than class **D**, and we found that one of them coincides with the set of allocations that are individually rational. Another set, $\mathbf{D}_1$, might be greater interest for games with empty class **D**, since it is always nonempty and no bigger than the set of imputations.

For PFF games, we were able to establish the nonemptiness of the analogous class $\overline{\mathbf{D}}$ for symmetric and partition-monotonic games. We also found small games having $\overline{\mathbf{D}}$

empty, which suggests that perhaps other classes of reasonable outcomes should be studied for these games. For example, we might consider extending the definition of $\mathbf{D}_s$ to PFF games, or we might define a class of outcomes for which the payoffs to the members of a fixed partition are reasonable. Also, concepts of rationality should be examined for PFF games that may be related to Milnor's, or other, notions of reasonableness.

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