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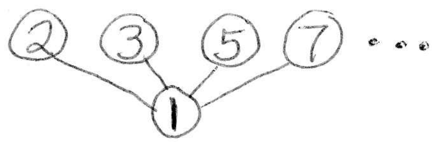
# The Non-Planarity of the Lattice Diagram of the Natural Numbers Under the Relation Divides

by Chellie Ramer

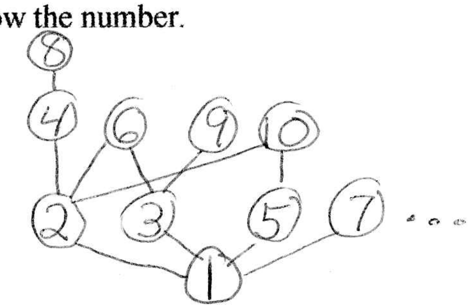
When presented with a classroom exercise of constructing a lattice diagram for the relation divides on the set {1, 2, 3,..., 15}, questions quickly arose about what the lattice diagram would look like for the entire set of natural numbers under the relation divides. We begin by defining the relation divides and constructing the lattice diagram for the relation.

**Definition**. Let n and m be natural numbers. We say that n *divides* m if there exists an integer a such that m=an. We denote this by n|m [1, p.140].

We can use the relation divides on the natural numbers to construct a lattice diagram. We start with one. This divides everything and so is a least element. The next row is the prime numbers. Each of these is connected to one.




From here, we add the rest of the natural numbers, connecting each additional number to all of its divisors on the row just below the number.

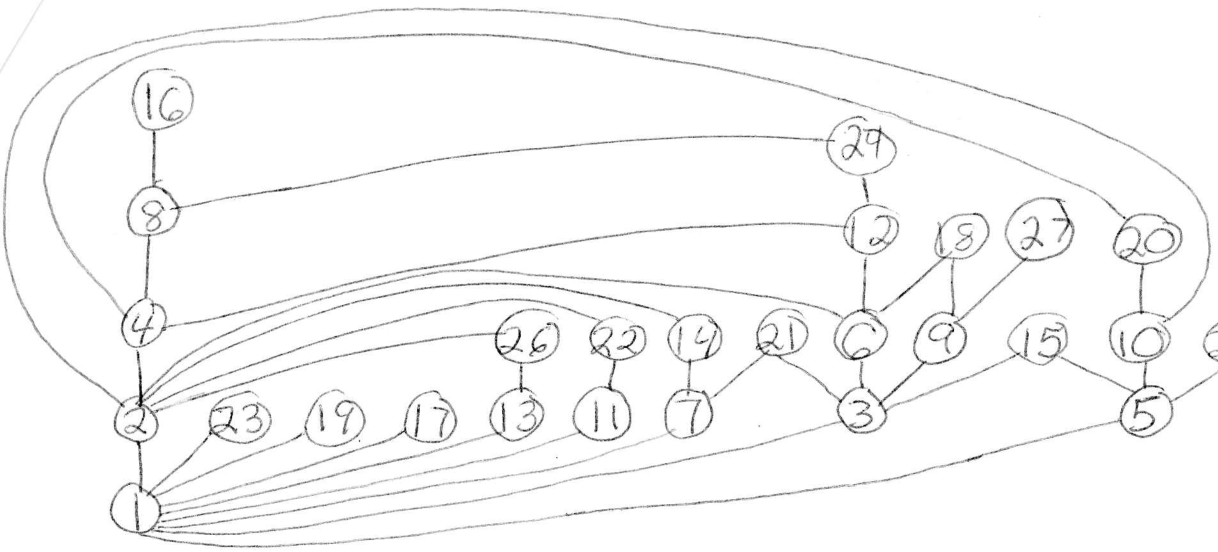



Each number must be put on the lowest level possible. In other words, each number belongs on the row just above its largest divisor. In this way we can construct a lattice diagram of the relation | on the set of natural numbers. Notice that in the above lattice diagram the line from ten to two crosses the lines from nine to three and from six to three. This leads to the natural question of does this always happen, or is there a way to draw the lattice diagram so that none of the lines cross?

**Definition**. A graph is said to be *planar* if it can be drawn in the plane in such a way that the edges do not cross [1, p.91].

**Theorem**. The lattice diagram that depicts the order | on the set {1, 2, 3,..., 27} is planar.

**Proof**. To show that a graph is planar, we need to show that there exists a way to draw the graph on paper, the plane, such that the drawing is planar. The following is one planar representation of the lattice diagram of the set {1, 2, 3,..., 27} under |.

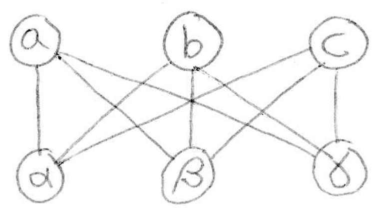



Thus, the lattice diagram depicting the relation | on the set {1, 2, 3,…, 27} is planar. □

It turns out that adding 28 to the set causes the lattice diagram to no longer be planar.

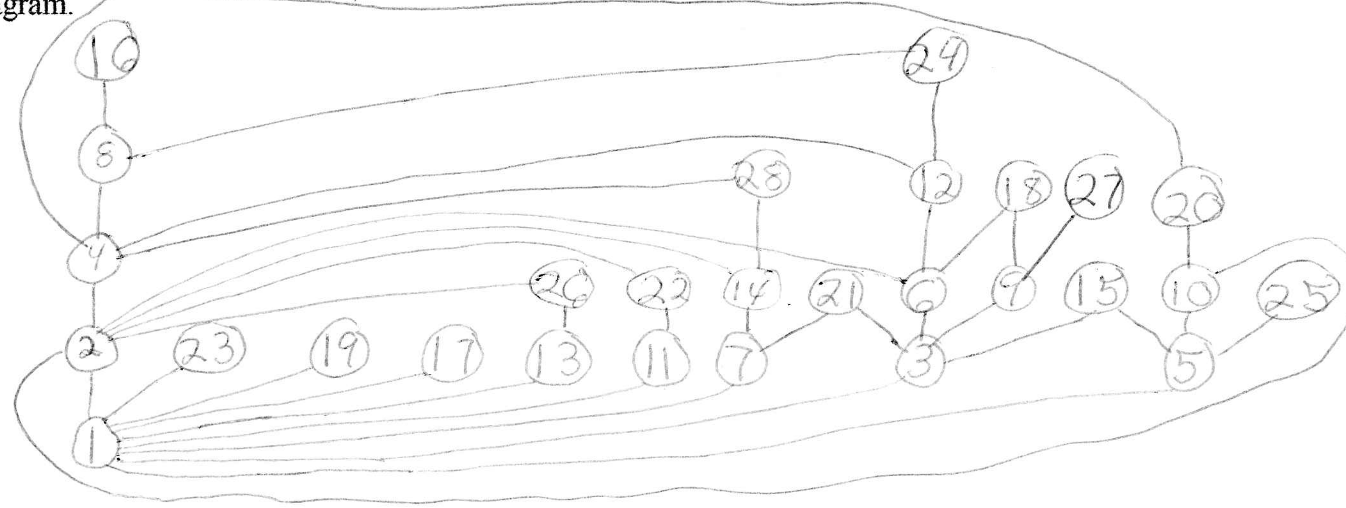
**Definition**. A graph G is *contractible* to a graph H if H may be obtained from G by edge contractions [**2**, p.99]. Let e=xy be an edge of G. We *contract* the edge e into a new vertex $v_e$, which becomes adjacent to all the former neighbors of x and y [**3**, p.16].

**Definition**. $K_{3,3}$ is defined as the lattice diagram constructed from two distinct sets of three distinct vertices each such that each vertex in the first set is connected to all the vertices in the second set.

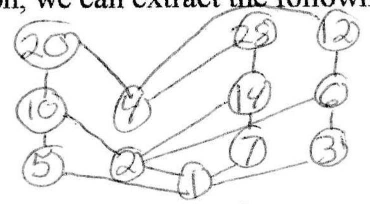



**Theorem.** The lattice diagram that depicts the order | on the set {1, 2, 3,…, 28} contains a sub-graph contractible to $K_{3,3}$.

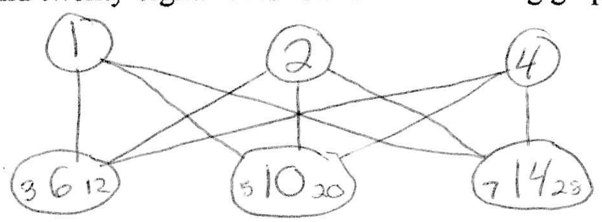
**Proof**. The lattice diagram of the relation | on the set {1, 2, 3,…, 28} is depicted in the following diagram.



From this graph, we can extract the following sub-graph.




This sub-graph is contractible. First, contract the edge from three to six into a new vertex; then we can contract the edge from this vertex to twelve creating a vertex that in essence contains the vertices three, six and twelve. This process can be repeated for five, ten and twenty and for seven, fourteen and twenty-eight. This forms the following graph, which is $K_{3,3}$.




Thus, the lattice diagram of the order | on the set {1, 2, 3,..., 28} contains a sub-graph that is contractible to $K_{3,3}$. □

**Wagner Criterion**. A graph is planar if and only if it has no sub-graphs contractible to $K_5$ or $K_{3,3}$ [**2**, p.99].

**Theorem**. The lattice diagram depicting the order | on any sub-set of the natural numbers containing the set {1, 2, 3,..., 28} is non-planar.

**Proof**. As shown in the above proof, the lattice diagram of the relation | on the set {1, 2, 3,..., 28} contains a sub-set contractible to $K_{3,3}$. By the Wagner Criterion, we know therefore that this lattice diagram is non-planar. Since adding vertices and edges to a non-planar graph would never be able to make the crossing lines in the non-planar graph no longer cross, we can say that any graph with a non-planar sub-graph is also non-planar. The lattice diagram of the order | on any sub-set of the natural numbers containing the set {1, 2, 3,..., 28} will by necessity have as a sub-graph the lattice diagram of the order | on the set {1, 2, 3,..., 28}. Therefore, the lattice diagram depicting the order | on any sub-set of the natural numbers containing the set {1, 2, 3,..., 28} is non-planar. □

**Corollary**. The lattice diagram that depicts the order | on the set of natural numbers is non-planar.

**References.**

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