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<u>The Shapley Value for Partition Function Form Games</u>

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Partition function form games were first introduced in Lucas and Thrall in 1963 as a generalized form of characteristic function form games. Both R.B. Myerson and E.M. Bolger have defined values on partition function form games. In this paper an extension of the Shapley value is sought which will satisfy linearity, efficiency, symmetry and dummy. Different axioms are then incorporated to place bounds on the various remaining parameters.

The following background information, along with 12 axioms and 7 definitions, is key to the paper.

<u>Background Information</u>:

$N = \{1,2,...,n\}$ is the set of players in a n-person game.

$CL = \{S \mid S \subseteq N, S \neq \emptyset\}$ is the set of coalitions of N.

$PT(S) = \{\{S^1,...,S^m\} \mid S^1 \cup ... \cup S^m = S, \forall j\ S^j \neq \emptyset, \forall k\ S^k \cap S^j = \emptyset$ if $k \neq j\}$ is the set of partitions of S.

$ECL = \{(S,P) \mid S \in CL, P \in PT(N-S)\}$ is the set of embedded coalitions, that is, the coalition S is faced with the players in N-S grouped by the partition P.

An n-player <u>game</u> in partition function form is any $W \in R^{ECL}$, that is, W is a function from embedded coalitions to real numbers. We may interpret $W(S;P)$ to be the amount S would receive if the players in N-S cooperated according to the partition P.

A <u>value</u> is a function Φ from some class of n-player games to R^n, the set of allocations. That is, $\Phi_i(W)$ is the allocation to player i in the game W.

A game is superadditive if any combination of two coalitions has a higher value than the two coalitions alone: $W(S;P\cup\{T\}) + W(T;P\cup\{S\}) \leq W(S\cup T;P)$ for all disjoint coalitions S and T and partitions $\{S,T\}\cup P$ of N. A game is coalition monotonic if adding players to a coalition, without moving other players, raises its worth: $W(S;P) \leq W(T; \{R-T: R\in P\})$, for all $S\subseteq T$. A game is partition monotonic if a coalition's worth is higher the less cooperation there is among the players outside the coalition: $W(S;P) \leq W(S;Q)$ whenever Q is a refinement of P, i.e., $R\in Q \Rightarrow \exists S\in P$ such that $R\subseteq S$.

We also define a partial order on embedded coalitions in the following manner: $(T;Q) \geq (S;P)$ if $S \subseteq T$ and Q is a refinement of $\{R-S: R\in P\}$. A unanimity game for $(S;P)$ is the game W defined by $W(T;Q) = 1$ if $(T;Q) \geq (S;P)$ and $W(T;Q) = 0$ otherwise.

Axioms and Definitions:

For the following axioms and definitions, let $S,T \subseteq N$, $P \in PT(N-S)$ and $Q \in PT(N-T)$, and W and V be games on N. The definitions are given as conditions which hold for all games in some class G. The reference to the class G has been omitted from each definition.

Definition 1: Suppose $\pi:N\rightarrow N$ is any permutation of the set of players. Then π acts as a permutation on CL and ECL in the following way:

$\pi(S) = \{\pi(j) \mid j\in S\}$, $\forall S\in CL$, and

$\pi(S^0, \{S^1,\ldots,S^k\}) = (\pi(S^0), \{\pi(S_1),\ldots,\pi(S^k)\})$,

$\forall(S^o, \{S^1,...,S^k\}) \in ECL$.

Symmetry: the payoff to a player does not depend on its name. Thus, $\forall j \in N$ and for every game W,

$\Phi_j(W) = \Phi_{\pi(j)}(\pi \circ W)$ where $\pi \circ W$ is the game that results from permutating the players in game W.

Linearity: $\Phi(aW+bV)=a\Phi(W)+b\Phi(V)$ for all real $a,b \in \mathbb{R}$.

Efficiency: $\Phi_1(W) + ... + \Phi_n(W) = W(N;\emptyset)$.

Definition 2: Player j is a dummy player if the value of any coalition S containing j does not change when j leaves it: $W(S;P)= W(S-\{j\};Q)$ for each $(S;P) \in ECL$ such that $j \in S$ and $|S| \geq 2$, where Q is the partition that results from adding j to one of the members of $P \cup \{\emptyset\}$ and $W(\{i\};P) = 0$ for $P \in PT(N-\{i\})$

Dummy Axiom: If a player does not contribute anything to any coalition, then that player should be allocated nothing. If j is a dummy in game W, then $\Phi_j(W)=0$.

Definition 3: Given $W \in R^{ECL}$ and $S \in CL$, S is a carrier of W when the members in S are the only part of the coalition that makes any difference in the value. S is a carrier iff $W(T_1;Q_1) = W(T_2;Q_2)$ whenever $T_1 \cap S = T_2 \cap S$, $Q_1|S = Q_2|S$, and $Q_1|(N-S) = Q_2|(N-S)$, where $Q|S = \{R \cap S: R \in Q \text{ and } R \cap S \neq \emptyset\}$

Carrier Axiom: $\forall(S;P) \in ECL$, if S is a carrier of W, then

$$\Sigma_{n \in S}\Phi_n(W) = W(N;\emptyset).$$

Since only the players in S give value to a coalition, only the members in S are allocated something.

Definitions 1 and the carrier axiom are from Myerson (1977).

Definition 5: The dummy extension of W is the game W^d defined on the player set $N \cup \{d\}$ by $W^d(S;P) = W(S-\{d\};\{R-\{d\}:R \in P\})$ $\forall$ ECLS (S:P). Dummy players are added to coalitions and partitions.

Dummy Independence: $\forall d \notin N$, and $\forall i \in N$, $\Phi_i(W^d) = \Phi_i(W)$. Adding dummy players does not affect the allocations to the other players.

Definition 5 and the dummy independence axiom are from Bolger (1987).

Group Monotonicity: if $W(S;P) \geq V(S;P)$ and $W(T;Q) = V(T;Q)$, $\forall (T;Q) \neq (S;P)$, then $\Phi_i(W) \geq \Phi_i(V)$, $\forall i \in S$.

If two games differ only in the values of a coalition S, then the members of S get a higher allocation in the game where their value is higher.

Complementary Group Monotonicity: if $W(S;P) \geq V(S;P)$ and $W(T;Q) = V(T;Q)$, $\forall (T;Q) \neq (S;P)$, then $\Phi_i(W) \leq \Phi_i(V)$, $\forall i \notin S$. In two games that differ only in the value of S, the players not in S will be allocated less in the game where S has a larger value.

<u>Definition 6:</u> In a game W, the marginals for player i are the quantities $W(S;P)-W(S-\{i\};Q)$, where $i \in S$ and Q is identical to P except that i has been added to one of the members of $P \cup \emptyset$.

Strong Monotonicity: if each marginal for player i on game W is greater than or equal to the corresponding marginal for player i on game V, then $\Phi_i(W) \geq \Phi_i(V)$.

Marginalist Axiom: if the corresponding marginals for player i are the same on two games W and V, then the allocation Φ_i should be the same for both games, $\Phi_i(W) = \Phi_i(V)$.

Strong Marginalist: let $i \in N$. If for each partition Q, where

$T \in Q$, $i \in T$ and the summations over all partitions P of N that can be obtained from Q by moving i from T into another, possibly empty, set we have $\Sigma[W(T;Q)-W(T-\{i\};P)] = \Sigma[V(T;Q)-V(T-\{i\};P)]$, then $\Phi_i(W) = \Phi_i(V)$.

Proposition 1. The value Φ is linear iff for some constant vector $b \in \mathbb{R}^n \times ECL$

$$\Phi_i(W)=\sum_{(S;P)\in ECL} b(i,S;P)w(S;P)$$

Proof. The formula is clearly linear. Conversely, suppose Φ is linear. Define $V^{S,P}$ by:

$V^{S,P}(S;P) = 1$

$V^{S,P}(T;Q) = 0$ for $(T;Q)\neq(S;P)$

Clearly, $\{V^{S,P} : (S;P) \in ECL\}$ forms a basis for $\mathbb{R}^{ECL}$ (comparable to the standard basis) In fact, $W \in \mathbb{R}^{ECL}$ implies that

$$W=\sum_{(S;P)\in ECL} w(S;P)v^{S,P}$$

By linearity,

$$\Phi_i(W)=\sum_{(S;P)\in ECL} w(S;P)\Phi_i(v^{S,P})$$

So we choose $b(S;P) = \Phi_i(V^{S,P})$ and the proof is complete.

Proposition 2. The value Φ is linear and satisfies the dummy axiom iff

$$\Phi_i(W) = \sum_{(S;P)\in ECL(i)} \sum_{R\in P\cup\{\emptyset\}} c(i,S;P;R)[w(S;P)-w(s-\{i\};P[i,R])] + \sum_{P\in PT(N-\{i\})} c(i,\{i\};P;\emptyset)w(\{i\};P)$$

for some constants c where ECL(i) is all (S;P) such that $i\in S$ and $|S|\geq 2$ and P[i,R] is the partition $P=\{R\}\cup\{R\cup\{i\}\}$.

Proof. Φ obviously satisfies the dummy axiom since the payoff to a player depends only on the marginals, which will be 0 when the player is a dummy.

Conversely, let $\bar{S}\subseteq N$, $|\bar{S}|\geq 2$, and $P\in PT(N-\bar{S})$. Define W such that $W(\bar{S};P)= W(\bar{S}-\{i\};P[i,R]) = 1 \ \forall \ R\in P\cup\emptyset$, and 0 otherwise. Then player i is a dummy player. We already have:

$$\Phi_i(W)=\sum_{(S;P)\in ECL} b(i,S;P)w(S;P)$$

a reordering of terms yields the expression

$$\Phi_i(W) = \sum_{(S;P)\in ECL(i)} [b(i,S;P)w(S;P) + \sum_{R\in P\cup\{\emptyset\}} b(i,S-\{i\};P[i,R])w(S-\{i\};P[i,R])]$$

$$= \sum_{(S;P)\in ECL(i)} [b(i,S;P) + \sum_{R\in P\cup\{\emptyset\}} b(i,S-\{i\};P[i,R])w(S;P) + \sum_{R\in P\cup\{\emptyset\}} -b(i,S-\{i\};P[i,R])[w(S;P)-w(S-\{i\};P[i,R])]]$$

Using the W defined above, this value becomes

$$\Phi_i(W)=b(i,S;P)=-\sum_{R\in P\cup\emptyset} b(i,S-\{i\};P[i,R])$$

and since i is a dummy,

$$b(i,S,P)=-\sum_{R\in P\cup\emptyset} b(i,S-i,P[i,R])$$

Renaming $-b(i,S-\{i\};P[i,R])$ with $c(i;S;P;R)$ for $|S|\geq @$ and $b(i,\{i\};P$ with $c(i,\{i\};P;\emptyset)$ we have shown that linearity and dummy imply the expression stated in the proposition. Since this formula depends only on the marginals of W, we have also proved that linearity and dummy imply marginalism.

Let $|S|$ denote the size of S and $\|P\|$ denote the sizes of the members of P, that is $\|P\|$ is the multiset $\{|R|:R\in P\}$. For example, if $P = \{ \{1,4\},\{2,3,6\},\{5,7\} \}$ then $\|P\| = \{2,3,2\}$.

<u>Proposition 3</u>. If Φ is satisfies linearity dummy, and symmetry, then

$$\Phi_i(W) = \sum_{(S;P)\in ECL(i)} \sum_{R\in P\cup\{\emptyset\}} d(|S|;\|P\|;|R|)[w(S;P)-w(S-i;P[i,R])]$$
$$+ \sum_{P\in PT(N-\{i\})} d(1;\|P\|;0)w(\{i\};P)$$

<u>Proof</u>. Define π to be a permutation such that $\pi(i_1) = i_2$

$$-c(i_1,;S_1;P_1;R_1)=\Phi_i(v^{S_1-i_1,P-1[i_1,R_1})=\Phi_{\pi(i_1}\pi v^{S_1-i_1,P_1[i_1,R_1]})=c(i_2;S_2;P_2;R-2)$$

and $\pi(S_1) = S_2$. Then

Thus the coefficients depend only on the sizes $|S|$, $\|P\|$, $|R|$. So the c's can be replaced with coefficients of the form $d(|S|;\|P\|;|R|)$.

The constants d have an interpretation as the power of "almost dummies" in certain simple games. Let $|S| = s$, $\|P\| = p$, and $|R| = r$. Define the game W by $W(T;Q) = 1$ whenever $(T;Q) \geq (S-\{i\}; P[i,R'])$ for some $R' \in P-\{R\} \cup \{\emptyset\}$, and $W(T;Q) = 0$ otherwise. Then player i is a dummy except in moving from S to R in the embedded coalition (S;P), and so it can be easily seen that the payoff to player i is $d(s;p;r)$.

Efficiency does not change this value, it simply puts a restriction on the coefficients as seen in the following theorem.

Theorem. If Φ satisfies linearity, dummy, symmetry and efficiency, then Φ is equal to the value in proposition three where the constants d satisfy the following recursion relation:

$$d(n;0;0) = \frac{1}{n}$$

$$d(s;p;0) = \sum_{r \in p} \left[\frac{r}{s} d(s+1;p-\{r\} \cup \{r-1\}) - d(s;p;r) \right]$$

Proof. Suppose $(\bar{S};P) \neq (N;\emptyset)$, $|\bar{S}| \geq 2$. Consider the game where $w(\bar{S};P) = 1$ and $w(S;P) = 0$ otherwise. By efficiency,

$$0 = \sum_{i \in N} \Phi_i(W) = \sum_{i \in \bar{S}} \sum_{R \in \bar{P} \cup \emptyset} d(|\bar{S}|;\|\bar{P}\|;|R|) + \sum_{i \notin \bar{S}} -d(|\bar{S}|+i;\|\bar{P}-i\|;|R-i|)$$

Since the term inside the first sum does not depend on i and for each R∈P, the term in the second sum does not depend on i,

$$=|\bar{S}|\sum_{R\in P\cup 0} d(|\bar{S}|;\|\bar{P}\|;|R|)-\sum_{R\in\bar{P}}|R|d(|\bar{S}|+1;\|\bar{P}\|-\{|R|\}\cup\{|R-1|\};|R|-1)$$

Now suppose s = |S|, r = |R| and p = ‖P‖. Then the above expression implies that

$$s\sum_{r\in p\cup 0} d(s;p;r)=\sum_{r\in p} rd(s+1;p-\{r\}\cup\{r-1\};r-1)$$

or

$$d(s;p;0=\sum_{r\in p}\frac{r}{s}d(s+1;p-\{r\}\cup\{r-1\})-d(s;p;r)$$

When |S| = 1, by efficiency we obtain:

$$0=d(1;\|P\|;0)+\sum_{R\in\bar{P}}-d(2;\|\bar{P}-i\|;|R-i|)$$

Simplifying the above expression in a manner analogous to the previous case we obtain:

$$d(1;p;0)=\sum_{R\in P} rd(2;p-\{r\}\cup\{r-1\})$$

It is also possible to prove these characteristics on the class of superadditive, partition and coalition monotonic games by replacing the games used in the above proof with their superadditive, partition and coalition monotonic covers and inducting on the ECL's starting with $(N;\emptyset)$ and in the direction of $\succeq$.

Thus the payoff to player 1 in a three player game using this class of values is:

$$\begin{aligned}\Phi_1(W) = {} & d(3;0;0)\ [W(123;\emptyset) - W(23;1)] \\ & + d(2;1;0)\ [W(12;3) - W(2;1,3)] \\ & + d(2;1;1)\ [W(12;3) - W(2;13)] \\ & + d(2;1;0)\ [W(13;2) - W(3;1,2)] \\ & + d(2;1;1)\ [W(13;2) - W(3;12)] \\ & + d(1;1,1;0)\ W(1;2,3) + d(1;2;0)\ W(1;23).\end{aligned}$$

Putting in the values of the d's the payoff becomes:

$d(3;0;0)=(1/3) \quad \Rightarrow \quad (1/3)[W(123,\emptyset)-W(23;1)]+$

$2[d(2;1;0)+d(2;1;1)]=d(3;0;0) \quad \Rightarrow \quad (1/6 - \mu_1)[W(12;3)-W(2;1,3)]+$

$\mu_1\ [W(12;3)-W(2;13)]+$

$(1/6 - \mu_1)[W(13;2)-W(3;1,2)]+$

$\mu_1\ [W(13;2)-W(3;12)]+$

$d(1;1,1;0)=2d(2;1;0) \quad \Rightarrow \quad (1/3 - 2\mu_1)\ W(1;2,3)+$

$d(1;2;0)=2d(2;1;1) \quad \Rightarrow \quad 2\mu_1\ W(1;23).$

Using the recursion relation, the payoff to player 1 in a four player game would appear as follows:

Equation: $\quad \Phi_1(W)=$

$d(4;0;0)=(1/4) \quad \Rightarrow \quad (1/4)[W(1234;\emptyset)-W(234;1)]+$

$3[d(3;1;0)+d(3;1;1)]=d(4;0;0) \quad \Rightarrow \quad (1/12-\mu_1)[W(123;4)-W(23;1,4)]+$

$\mu_1\ [W(123;4)-W(23;14)]+$

$$(1/12-\mu_1)[W(124;3)-W(24;1,3)]+$$

$$\mu_1\ [W(124;3)-W(24;13)]+$$

$$(1/12-\mu_1)[W(134;2)-W(34;1,2)]+$$

$$\mu_1\ [W(134;2)-W(34;12)]+$$

$2[d(2;2;2)+d(2;2;0)]=2d(3;1;1)\ \Rightarrow$

$$(\mu_1-\mu_2)[W(12;34)-W(2;34,1)]+$$

$$\mu_2\ [W(12;34)-W(2;134)]+$$

$$(\mu_1-\mu_2)[W(13;24)-W(3;24,1)]+$$

$$\mu_2\ [W(13;24)-W(3;124)]+$$

$$(\mu_1-\mu_2)[W(14;23)-W(4;23,1)]+$$

$$\mu_2\ [W(14;23)-W(4;123)]+$$

$2[2d(2;1,1;1)+d(2;1,1;0)]=2d(3;1;0)\ \Rightarrow$

$$(1/12-\mu_1-2\mu_3)[W(12;3,4)-W(2;1,3,4)]+$$

$$\mu_3\ [W(12;3,4)-W(2;13,4)]+$$

$$\mu_3\ [W(12;3,4)-W(2;14,3)]+$$

$$(1/12-\mu_1-2\mu_3)[W(13;2,4)-W(3;1,2,4)]+$$

$$\mu_3\ [W(13;2,4)-W(3;12,4)]+$$

$$\mu_3\ [W(13;2,4)-W(3;14,2)]+$$

$$(1/12-\mu_1-2\mu_3)[W(14;2,3)-W(4;1,2,3)]+$$

$$\mu_3\ [W(14;2,3)-W(4;12,3)]+$$

$$\mu_3\ [W(14;2,3)-W(4;13,2)]+$$

$d(1;3;0)=3d(2;2;2)\ \Rightarrow\ 3\mu_2\ [W(1;234)]+$

$d(1;2,1;0)=d(2;2;0)+2d(2;1,1;1)\ \Rightarrow\ (\mu_1-\mu_2+2\mu_3)[W(1;23,4)+W(1;24,3)$

$+W(1;34,2)]+$

$d(1;1,1,1;0)=3d(2;1,1;0)\ \Rightarrow\ (1/4-3\mu_1-6\mu_3)[W(1;2,3,4)].$

Where a bound on the μ's is obtained by requiring that the coefficients be positive. This is not unreasonable since the coefficients are multiplying marginals and the higher the marginals of a player the more that player should receive in the final allocation