# Auctions

A Senior Comprehensive Project
by
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I hereby recognize and pledge to fulfill my responsibilities, as defined in the Honor Code, and to maintain the integrity of both myself and the College community as a whole.

Pledge:

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Jennifer L. Timcho

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## Abstract

This is a study of the history and types of auctions held around the world. The single item first-price, sealed bid auction and the multiple object auction were mathematically modeled with the goal of finding the optimum bidding strategy in each case.

# 1 Chapter One: The History and Types of Auctions

## 1.1 History

The first to mention auctions throughout history is Herodotus, a Greek historian. He tells us that around 500 BC the Babylonians held auctions every year in their villages. Once a year all of the village maidens were brought together to be auctioned off to husbands. The men stood around the maidens as they were called up one at a time and offered for sale, beginning with the most beautiful. When she was sold, the next most beautiful maiden was brought up. The rich Babylonian men would bid against one another for the attractive women, and they would be sold for a very high price. After all of the beautiful women had been bid on, the ugliest woman would then be called up. The poorer Babylonian men would bid to accept her for an amount of money. Whoever would marry her for the smallest amount of money, she was assigned to. These payments to the poorer men for taking an ugly wife were made from the money that the rich men had paid for the most beautiful women.[3, page 44]

The Romans also had auctions in very early times; they held them for commercial trade. To pay for their debts, Romans would sometimes auction off their furniture or other belongings.[5, page 7] On the Greek island of Delos, the legendary birthplace of the god Apollo, slave auctions were held in which Romans, Greeks, and pirate traders would attend.[5, page 8] The largest auction of all time took place approximately 93 AD. This was during the Roman supremecy, and the entire Roman empire was put up for sale by the Praeterian Guard. Emperors bid on the empire in the hopes of expanding their kingdoms.[5, page 9]

There is a lack of information about auctions after the fall of Rome and throughout the dark ages. The earliest evidence of auctions after this period is in France, about 1556. Auctions were held on the property of a deceased owner or a debtor.[5, page 15]

"The earliest reference to the word auction in the *Oxford English Dictionary* dates from 1595"[5, page 16], but there is evidence that auctions were going on in England prior to this. Evidence submitted in the London courts during a case in 1795 shows that auctions were held in England at the end of the fifteenth century. It is documented that Henry VII had a definite definition of an auctioneer, but it is not until late in the seventeenth century that we find hard evidence of auctions being conducted. A record from this time period indicates that Conditions of Sale in auctions were already set and that at least three different types of bidding were in use.[5, page 16] It is clear that by 1682 auctions were very common events in London. An issue of the London Gazette during this year makes reference to "the daily attendence at the auction house."[5, page 20]

It is common knowledge that slave auctions were held in early America. However, information about specific sales of slaves is hard to find before the 1700's. In 1736 there is a record of 300 African slaves being sold at an auction in Yorktown, Virginia. In 1737 in Yorktown, there is record of 490 slaves sold at an auction. These slave auctions finally ended in 1865, with the end of the Civil War.[5, page 31]

At the same time slave auctions were being held in America, auctions were becoming popular in Europe. During the latter part of the eighteenth century, the auction business was growing rapidly in London and Paris. These auctions were moving towards the types of auctions that are held today. In 1745 Samuel Baker had his first auction sale of books. He would later found Sotheby's, a famous auction house in England. The following, taken from Learmount's book "A History of the Auction," is a list of sophisticated conditions for auctions set up by Samuel Baker:

1. That he who bids most is the Buyer, but if any Dispute arises, the Book or Books to be put to sale again.
2. That no Person advances less than Sixpence each bidding, and after the Book arises to One Pound, no less than One Shilling.
3. The Books are in most elegant Condition, and supposed to be Perfect, but if any appear otherwise before taken away, the Buyer is at his Choice to take or leave them.
4. That each Person give in his Name, and pay Five Shillings in the Pound (if demended) for what he Buys, and that no Book be deliver'd in Time of Sale.
5. The Books must be taken away at the Buyer's Expence, and the Money paid at the Place of Sale, within Three Days after each Sale is ended. Any Gentleman who cannot attend the Sale, may have their Commissions receiv'd and faithfully executed By their most Humble servent Samuel Baker.[5, pages 47–48]

Before the war of 1812 the American government discouraged American ships from entering European ports, shutting off most of the foreign supply. This brought about a great demand for goods, especially the mass produced goods from Britain. British merchants stocked their possessions closest to

America, such as Bermuda and Halifax, with such goods. They waited there for the hostility to cease, and when it did, they sold their goods by auction. They chose this method of sale because it was the most convenient and the quickest. The domestic merchants complained about the mass dumping of British goods on the American market, but a campaign developed against the autioneers.[5, page 82] The claim was that these auctions were destroying local trade and disturbing commerce and industry in general.

An anti-auction movement began in America, and the first phase of this lasted until 1824. Boycott was one method used against auctioneers; this proved to be unsuccessful. In 1828 Congress was flooded with petitions concerning the outrages committeed by auctioneers.[5, page 83] This was not a one sided argument. There were also statements submitted to Congress about the honor and honesty of auctioneers.

Such periods of anti-auction sentiment coincided with recessions. Domestic merchants placed blame on the auctions for their poor trade. Today there are similar criticisms of auctions, that some are corrupt and rigged. These criticisms of auctions are voiced regularly, and just as in the past they eventually disappear.[5, page 99]

Today, auctions are held all over the world in various markets. In 1994, in the United States, the Federal Communications Commission (FCC) held its first auction for the licenses for wireless phone systems. The FCC worked with John McMillan, an expert in game theory, to determine how they could make the most money through an auction of these licenses. The potential bidders of this auction were companies such as Pacific Bell, Bell Atlantic Corp., and MCI Communications Corp. These companies also hired game theorists to argue with the FCC about how the auction should be run, and to help them bid strategically.[6, page 48]

Prior to July of 1994 the Federal Communications Commission decided which companies had access to available radio-spectrum bands. Licenses for these bands were auctioned off for the first time on July 25, 1994. This auction was almost completely designed by game theory experts. The FCC had to take into consideration that a well designed auction must account for certain uncertainties.

1. The seller does not know how much the bidders value what they are bidding on. In this case, if the government knew how much bidders valued the radio-spectrum bands, they could just set a price and there would be no need for an auction.
2. The bidders don't know how profitable it would be to win.
3. The bidders don't know how much the others think the item being auctioned is worth.[9, page 70]

## 1.2 Bidding Methods

In some auctions only one item is up for bid. This type of auction is known as a single item auction. There are various methods of bidding associated with these auctions, the most common of which is the ascending bid auction. Our word 'auction' comes from the Latin 'auctio', meaning increase. Therefore auctions are perceived to be run by bidders giving successive offers of increasing amounts.[5, page 6] This type of auction has been used in England for so long that it is sometimes referred to as the 'English' method.[5, page 127] There are ascending bid auctions held with different rules than the English method. One of these is the 'sale by candle'. This auction type was used for a long time in England. An inch of candle was set up, and when it was lit, the auctioneer would begin to accept ascending bids. The person to give the last bid before the candle went out would win the lot.[5, page 17] An auction very similar to the sale by candle is the sale by sand glass. The last bid to be called out before all of the sand in an hour glass runs out is accepted.[5, page 128]

Another type of auction, the descending bid, has been used in Holland for a very long time. For this reason this type of auction is sometimes referred to as the 'Dutch' auction. In this method of auction, the auctioneer begins by choosing a starting price greater than what he expects to make. He then calls out lower prices until the object is sold. Today this method of auction is used in Holland, England, and Israel in fish markets. In Holland bidders gather in amphitheaters to purchase cut flowers. Everyone sits at a desk facing a clock-type mechanism. The hand begins to rotate around the "clock" pointing to the highest selling price first, and as it rotates counterclockwise it points to lower and lower prices. Whenever a bidder hits a button at his desk, it stops the clock and the lot is sold to that bidder at the amount the pointer indicates.[5, page 130]

There are also second price auctions. In this case, the person who bids the highest amount wins, but he only pays the amount of the second highest bid. This auction type is sometimes called the 'Vickery auction' because Vickery showed that the best bidding strategy to use in an auction of this type is to bid what you are willing to pay.[4, page 65]

A sealed bid auction is one in which each bidder writes down the amount of his bid, seals his bid and gives it to the auctioneer. The auctioneer then opens the bids one by one and reads them aloud. The highest bid wins. Just as the ascending bid auction had different rules for how the bidding was done, the sealed bid auction has different methods. One type that originated in China is known as the handshake auction. The auctioneer has a piece of cloth covering his hand, and one by one the bidders shake his hand underneath the cloth, indicating by pressing the auctioneer's fingers the amount of their bid. After everyone makes an offer, the auctioneer announces the highest bidder. Until relatively recently this bidding method was used in Pakistan for the sale of dried fish.[1, pages 71–72] The whispered-bid auction is very much like the handshake auction. Bidders whisper their bids to the auctioneer one by one, and after each takes his turn the auctioneer announces a winner. This bidding method is used in the fish markets of Singapore, Manila, Venice, and Chioggia.[1, page 73]

When more than one object is up for bid, the auction is known as a multiple object auction. For this type of auction, the bidding methods for single item auctions may be used as each item is put up for bid one by one.

In the case where the Federal Communications Commission held an auction for licenses for wireless phone systems, they had to try to determine which rules of auction to use. The English auction, the Dutch auction, the sealed-bid auction, and the Vickery auction were all under consideration. The FCC had to choose which of these auction rules would best meet the goals of collecting a large amount of money, encourage competition, and create service areas that are efficient. If they chose the English auction, prices may go up as the participants can see what their rivals are doing; however, bidders may collude in order to keep the prices low. If the Dutch auction was used, the bidders may have been very cautious, resulting in lower bids. A sealed-bid auction would have less chance of collusion among the bidders, but this could also be less revenue for the government as bidders often act cautiously with this bidding scheme. The Vickery auction would have resulted in high bids, but the seller may end up getting too little.[6, page 48]

The multiple object auctions may also be held with people bidding simultaneously on the objects for sale. Simultaneous bidding is yet another method of auction. The United States Department of the Interior holds multiple object auctions with simultaneous bidding for the leasing of land owned or controlled by the government for oil and gas exploration and development. A list is posted by each state of all the available land for leasing. Bidders then makes an offer by submitting a bid on an "Automated Simultaneous Oil and Gas Lease Application" during the filing period. This filing period ends fifteen working days after the lists are posted. Once the filing period is closed, one applicant is selected randomly by computer for each parcel of land. If the applicants bid is not considered by the government to be high enough, a reselection is made from the remaining applicants.[8]

The FCC auction of radio-spectrum bands was held in multiple rounds. In each round the bids were sealed. This style limits the threat of bidders colluding. It also minimizes the chance of a winner's curse, which is when a bidder wins an auction by bidding very high, but then is unable to gain enough from the object to make his bid worthwhile.[9, page 70]

## 1.3 Common and Private Value Auctions

A common value auction is one in which all of the bidders are trying to estimate the value of what they are bidding on. The object that is up for bid would be worth the same amount to each bidder but none of the bidders know what the worth is. Suppose, for example, that a tract of land which may contain oil is up for bid. All of the bidders can estimate how much the land is worth, but no one will know the true value of the land in this case until years after the auction is held.[4, page 61]

In a private value auction each bidder knows how much he values the object being auctioned. However, he only has probabilistic information about how much the other bidders value that same object. In a private value auction the problem is not determining how much the object is worth. The only problem is determining how to bid strategically.

# 2 Chapter Two: Modeling a Single Item Private Value Auction

## 2.1 Set-up

We will now study the single item private value first-price, sealed bid auctions by studying the symmetric case. Suppose that for an auction we have a set $S$ of $n$ bidders, or players. So $S = \{1, 2, \ldots, n\}$. Each player knows what the object is worth to him, but he does not know how much it is worth to the other bidders. Each player places a nonnegative bid on the object without knowing the other bids. The highest bidder wins the object and pays the amount of his bid. Let $X_1, X_2, \ldots, X_n$ be random variables.

**Definition 1.** If $S$ is a sample space with a probability measure and $X$ is a real-valued function defined over the elements of $S$, then $X$ is called a *random variable.*[2, page 83]

Each of these $X_i$'s represents the value of the object being bid on by that player. For example, $X_1$ represents player one's value of the object being bid on. We assume that the $X_1, X_2 \ldots, X_n$ are independently and identically distributed with a distribution function G. So we have $G(x) =$ Prob $[X_i \leq x]$ for all $1 \leq i \leq n$.

**Definition 2.** The function $F$ defined by $F(X) = P[X \leq x]$ for $-\infty < x < \infty$ is called the *distribution function* of $X$.

To say that the random variables are independent is to say that the value of one of these variables does not depend probablistically on the value of any of the others.

**Definition 3.** $X_1, X_2, \ldots, X_n$ are *independent* random variables if $\text{Prob}[a_1 \leq X_1 \leq b_1, a_2 \leq X_2 \leq b_2, \ldots, a_n \leq X_n \leq b_n] = \text{Prob}[a_1 \leq X_1 \leq b_1] \cdot \text{Prob}[a_2 \leq X_2 \leq b_2] \cdots \text{Prob}[a_n \leq X_n \leq b_n]$ for all $a_1, a_2, \ldots, a_n, b_1, b_2, \ldots b_n$.

**Definition 4.** A *density function* is the derivative of the distribution function.

Assume our distribution function $G$ has a density function $g$. The density function indicates how likely a particular outcome is to occur. This assumption implies that Prob $[X_i \leq x] =$ Prob $[X_i < x]$ for all $x$. We will assume that $G(0) = 0$, or that the object is never worth a negative amount. Finally, we will make one rather technical assumption. That is that the density function $g$ is continuous at all but finitely many places. The reason for this last assumption will become apparent later.

Let $\mathbb{R}^+$ be the set of nonnegative real numbers. The strategy for player $i$ is given by $\sigma_i : \mathbb{R}^+ \to \mathbb{R}^+$ where $\sigma_i$ maps each player's nonnegative real number values to real bids. This strategy function is necessary because a player does not bid the amount he thinks the object is worth. The player must take into consideration what he thinks the other players' values are, and how accurate his estimated bid is. Each player wants to choose a bidding strategy that will give him the maximum expected payoff given the bidding strategies of all of the other players. If all players choose a strategy in this manner, the resulting profile of strategies is a *Bayesian equilibrium.*

Before doing any calculations, we make a few assumptions. We wish to find a Bayesian equilibrium, $(\sigma_1, \sigma_2, \ldots, \sigma_n)$. So, we begin by assuming that $(\sigma_1, \sigma_2, \ldots, \sigma_n)$ is in fact a Bayesian equilibrium. Also, we assume that

$\sigma_1, \sigma_2, \ldots, \sigma_n$ are all increasing functions. That is, each player would bid more for the object the more he values it. We will assume that $\sigma_1, \sigma_2, \ldots, \sigma_n$ are all differentiable functions. Also, $\sigma_1 = \sigma_2 = \cdots = \sigma_n = \sigma$. That is, all players use the same strategy.

## 2.2 The Expected Payoff Function

Now consider player $i$. Suppose that $X_i = x$, player $i$ bids $b$, and player $j$ uses strategy $\sigma_j$ for each $j \neq i$. The payoff for player $i$ is $(x - b)$ if $i$'s bid is the highest, and zero otherwise. Player $i$'s bid, $b$, is the highest if $b > \sigma_j(X_j)$ for all $j \neq i$. Hence, the expected payoff function for player $i$ is given by:

$$\pi_i(b, x, \{\sigma_j\}_{j \neq i}) = (x - b)\text{Prob}[b > \sigma_j(X_j) \quad \forall j \neq i].$$

We have assumed $\sigma_j$ is an increasing function, and this implies that $\sigma_j$ is one-to-one. So we know that $\sigma_j^{-1}$ exists. Taking $\sigma_j^{-1}$ of each side of the inequality in the probability, we now have:

$$\pi_i(b, x, \{\sigma_j\}_{j \neq i}) = (x - b)\text{Prob}[\sigma_j^{-1}(b) > X_j \quad \forall j \neq i].$$

To make notation less confusing, let $\gamma_j = \sigma_j^{-1}$.

$$\pi_i(b, x, \{\sigma_j\}_{j \neq i}) = (x - b)\text{Prob}[\gamma_j(b) > X_j \quad \forall j \neq i].$$

Because of the independence of our random variables $X_1, X_2, \ldots, X_n$, we may now write:

$$\pi_i(b, x, \{\sigma_j\}_{j \neq i}) = (x - b)\Pi_{j \neq i}\text{Prob}[\gamma_j(b) > X_j].$$

Putting our probability distribution function into the equation we have:

$$\pi_i(b, x, \{\sigma_j\}_{j \neq i}) = (x - b)\Pi_{j \neq i}G(\gamma_j(b)).$$

Because we assumed all $\sigma_i$'s are equal to $\sigma$, all $\gamma_j$'s are also equal to $\gamma = \sigma^{-1}$, and we may write:

$$\pi_i(b, x, \{\sigma_j\}_{j \neq i}) = (x - b)G(\gamma(b))^{n-1}.$$

## 2.3 Finding the Optimum Strategy

By taking the derivative of the above equation with respect to $b$ we will be able to determine the optimum bid. So we begin by using the product rule and the chain rule:

$$\frac{\partial \pi_i}{\partial b} = (x - b)(n - 1)G(\gamma(b))^{n-2}g(\gamma(b))\gamma'(b) + (-1)G(\gamma(b))^{n-1}.$$

We wish to maximize our payoff function, $\pi_i$. Since $(\sigma_1, \sigma_2, \ldots, \sigma_n)$ is a Bayesian equilibrium, the payoff function is maximized when the bid, $b$, is equal to $\sigma(x)$. Since we are trying to find the optimum bid, we now substitute $\sigma(x)$ into the above equation for $b$, and set it equal to zero. That is

$$\frac{\partial \pi_i}{\partial b} |_{b=\sigma(x)} = 0$$

$$(x - \sigma(x))(n-1)G(\gamma(\sigma(x)))^{n-2} g(\gamma(\sigma(x)))\gamma'(\sigma(x)) = G(\gamma(\sigma(x)))^{n-1}.$$

Recall that $\gamma$ is the inverse of $\sigma$, so the composition of these two functions will be the identity function, and the above will simplify to:

$$(x - \sigma(x))(n-1)G(x)^{n-2} g(x)\gamma'(\sigma(x)) = G(x)^{n-1}.$$

Remember that $\gamma = \sigma^{-1}$. So $\gamma'(\sigma(x)) = (\sigma^{-1})'(\sigma(x))$. And recall from calculus that the derivative of the inverse of a function is equal to the reciprocal of the derivative of the original function. So we can substitute $\frac{1}{\sigma'(x)}$ in the above equation for $\gamma'(\sigma(x))$. We now have:

$$(x - \sigma(x))(n-1)G(x)^{n-2} g(x)\frac{1}{\sigma'(x)} = G(x)^{n-1}.$$

Now simplify.

$$(x - \sigma(x))(n-1)G(x)^{n-2} g(x) = G(x)^{n-1}\sigma'(x)$$

$$x[(n-1)G(x)^{n-2} g(x)] = G(x)^{n-1}\sigma'(x) + \sigma(x)[(n-1)G(x)^{n-2} g(x)]$$

Notice that in the above equation, the right hand side is equal to $\frac{d}{dx}[G(x)^{n-1}\sigma(x)]$. So we can now write:

$$x[(n-1)G(x)^{n-2} g(x)] = \frac{d}{dx}[G(x)^{n-1}\sigma(x)].$$

Before integrating both sides of this equation, change all of the $x$'s to $t$'s.

$$t[(n-1)G(t)^{n-2}g(t)] = \frac{d}{dt}[G(t)^{n-1}\sigma(t)].$$

Now we may integrate with respect to $t$ from 0 to $x$:

$$\int_0^x t(n-1)G(t)^{n-2}g(t)\,dt = G(x)^{n-1}\sigma(x) - G(0)^{n-1}\sigma(0)$$

Recall that we assumed that $G(0) = 0$, so the second term in the right hand side of the previous equation is just 0. Solving for $\sigma(x)$ we find that

$$\sigma(x) = \frac{(n-1)}{G(x)^{n-1}} \int_0^x tG(t)^{n-2}g(t)\,dt.$$

We must now check to see if this $\sigma$ fulfills all of the assumptions that we made on it at the very beginning. So we must check to be sure that the $\sigma(x)$ that we found is defined, differentiable with respect to $b$, increasing, and maximizes $\pi_i(b, x, \{\sigma_j\}_{j\neq i})$. Our $\sigma$ is defined for $x > 0$ because the second term is in fact integrable, this comes from the following Theorem:

**Theorem 1.** *If $f$ is a piecewise continuous function or a bounded monotonic function on $[a, b]$, then $f$ is integrable on $[a, b]$.[7, page 196]*

So our $\sigma(x)$ is defined as long as our density function $g$ is continuous at all but a finite number of places. In fact our density function $g$ does fulfill this requirement because of the assumption that we made on it at the very beginning of our calculations. This is precisely why we had to make such an assumption about the density function from the very beginning.

We know that our $\sigma(x)$ is differentiable because G is differentiable, and the Fundamental Theorem of Calculus allows us to differentiate the integral.

**Theorem 2.** *Let $f$ be an integrable function on $[a, b]$. For $x \in [a, b]$, let $F(x) = \int_a^x f(t)\,dt$. Then $F$ is continuous on $[a, b]$. If $f$ is continuous at $x_0$ in $(a, b)$, then $F$ is differentiable at $x_0$ and $F'(x_0) = f(x_0)$.[7, page 199]*

So $\sigma$ is differentiable everywhere that $g$ is continuous.

The product rule and the chain rule tell us that the product and composition of differentiable functions is differentiable. So by using the product and chain rules, the quotient rule, and the Fundamental Theorem of Calculus we can find the derivative of our $\sigma$.

$$\sigma'(x) = -\frac{(n-1)^2 G(x)^{n-2} g(x)}{(G(x)^{n-1})^2} \int_0^x tG(t)^{n-2} g(t)\,dt + \frac{(n-1)}{G(x)^{n-1}}[xG(x)^{n-2} g(x)].$$

We now check to be sure our $\sigma(x)$ is an increasing function. We do this by checking to see if $\sigma'(x) > 0$. We must simplify the $\sigma'(x)$ equation in order to do this. We begin with

$$\sigma'(x) = -\frac{(n-1)^2 G(x)^{n-2} g(x)}{(G(x)^{n-1})^2} \int_0^x tG(t)^{n-2} g(t)\,dt + \frac{(n-1)}{G(x)^{n-1}}[xG(x)^{n-2} g(x)].$$

Now put one of the $(n-1)$ quantities of the first term inside the integral.

$$\sigma'(x) = -\frac{(n-1) G(x)^{n-2} g(x)}{(G(x)^{n-1})^2} \int_0^x (n-1)tG(t)^{n-2} g(t)\,dt + \frac{(n-1)}{G(x)^{n-1}}[xG(x)^{n-2} g(x)].$$

We can calculate $\int_0^x (n-1)G(t)^{n-2} g(t)t\,dt$ using integration by parts.

Let $u = t$, $du = dt$, $dv = (n-1)G(t)^{n-2}g(t)\,dt$, and $v = G(t)^{n-1}$. This gives us:

$$\int_0^x (n-1)G(t)^{n-2}g(t)t\,dt = xG(x)^{n-1} - \int_0^x G(t)^{n-1}dt.$$

Putting this back into our $\sigma'(x)$ equation, we have:

$$\sigma'(x) = -\frac{(n-1)G(x)^{n-2}g(x)}{(G(x)^{n-1})^2}\left[xG(x)^{n-1} - \int_0^x G(t)^{n-1}\,dt\right] + \frac{(n-1)}{G(x)^{n-1}}\left[xG(x)^{n-2}g(x)\right].$$

Multiplying both the first and second terms out:

$$\sigma'(x) = -\frac{(n-1)G(x)^{n-2}g(x)xG(x)^{n-1}}{(G(x)^{n-1})^2} + \frac{(n-1)G(x)^{n-2}g(x)}{(G(x)^{n-1})^2}\int_0^x G(t)^{n-1}\,dt + \frac{(n-1)xG(x)^{n-2}g(x)}{G(x)^{n-1}}.$$

Simplifying:

$$\sigma'(x) = -\frac{(n-1)g(x)x}{G(x)} + \frac{(n-1)g(x)}{G(x)^n}\int_0^x G(t)^{n-1}\,dt + \frac{(n-1)xg(x)}{G(x)}.$$

The first term is the negative of the third term, so these two drop out of the equation. We are left with just

$$\sigma'(x) = \frac{(n-1)g(x)}{G(x)^n}\int_0^x G(t)^{n-1}\,dt$$

This term is positive because $G(x)$ and $g(x)$ are both positive for $x > 0$, so $\sigma'(x) > 0$. Therefore $\sigma$ is an increasing function.

We know that the $\sigma$ that we found is a maximum. This is because the payoff function $\pi_i(b, x, \{\sigma_j\}_{j \neq i}) = (x - b)G(\gamma(b))^{n-1}$ is equal to zero when the bid, $b$, is either zero or $x$, but is positive for any other bid $b \in (0, x)$. We found our optimum bid $\sigma$ by setting the derivative of the payoff function equal to zero and solving. Therefore, our $\sigma$ must give a maximum, because it cannot be a minimum. This is true assuming that $\sigma(x) \in (0, x)$. Our $\sigma(x)$ has this property, as shown below.

First recall the definition of $\sigma(x)$.

$$\sigma(x) = \frac{(n-1)}{G(x)^{n-1}} \int_0^x tG(t)^{n-2} g(t)\, dt$$

Again, put the $(n-1)$ inside the integral and integrate by parts. Doing this we will have:

$$\sigma(x) = \frac{1}{G(x)^{n-1}} \left[ xG(x)^{n-1} - \int_0^x G(t)^{n-1}\, dt \right]$$

Multiplying through, we will have:

$$\sigma(x) = \frac{xG(x)^{n-1}}{G(x)^{n-1}} - \frac{1}{G(x)^{n-1}} \int_0^x G(t)^{n-1}\, dt$$

First simplify:

$$\sigma(x) = x - \frac{1}{G(x)^{n-1}} \int_0^x G(t)^{n-1}\, dt$$

It is from here that we can see that our $\sigma(x) \in (0, x)$. The second term is positive, but as we will see, it is less than $x$. We can see this by bringing the $\frac{1}{G(x)^{n-1}}$ inside the integral.

$$\int_0^x \left(\frac{G(t)}{G(x)}\right)^{n-1} dt.$$

This integrand is less than one because here, $t$ will be less than $x$, which implies that $G(t) < G(x)$. Integrating $\left(\frac{G(t)}{G(x)}\right)^{n-1}$ and 1 from zero to $x$, we will see that the integral is less than $x$. So we have $x$ minus a positive number that is less then $x$, giving us $\sigma(x) \in (0, x)$.

## 2.4 Finding the Expected Payoff and the Expected Selling Price

Now that we have found our optimum bidding strategy there are several other values that we may calculate. We can find what our expected payoff will be. The payoff function that we formulated at the beginning of the chapter, $\pi_i(\sigma(x), x, \{\sigma_j\}_{j \neq i})$ was:

$$\pi_i(\sigma(x), x, \{\sigma_j\}_{j \neq i}) = (x - \sigma(x))(G(\gamma(\sigma(x)))^{n-1}.$$

Simplifying, we have,

$$\pi_i(\sigma(x), x, \{\sigma_j\}_{j \neq i}) = (x - \sigma(x))G(x)^{n-1}.$$

Now putting our optimum bidding strategy in for $\sigma(x)$, we have our expected payoff.

$$\pi_i(\sigma(x), x, \{\sigma_j\}_{j \neq i}) = \left(x - \frac{(n-1)}{G(x)^{n-1}} \int_0^x tG(t)^{n-2} g(t)\, dt\right) G(x)^{n-1}.$$

We can also find what the expected selling price of an object will be if all of the players are using this strategy. The selling price for this type of auction will be the amount of the highest bid. So the selling price, $P$, is a random variable, and the expected selling price is

$$E[P] = \max \{\sigma(X_1), \sigma(X_2), \ldots, \sigma(X_n)\}.$$

Let $H$ be the distribution function of the maximum, $Y$, of the bids. So $Y = \max \{\sigma(X_1), \sigma(X_2), \ldots, \sigma(X_n)\}$. Then we have:

$$H(y) = \text{Prob } [Y \leq y].$$

This is equal to the probability that each bid is less than or equal to $y$. Now we have:

$$H(y) = \text{Prob } [\sigma(X_1), \sigma(X_2), \ldots, \sigma(X_n) \leq y].$$

This can be broken into $n$ smaller probabilities because of our assumption of independence:

$$H(y) = \text{Prob } [\sigma(X_1) \leq y] \text{ Prob } [\sigma(X_2) \leq y] \ldots \text{ Prob } [\sigma(X_n) \leq y]$$

$$H(y) = \text{Prob } [X_1 \leq \gamma(y)] \text{ Prob } [X_2 \leq \gamma(y)] \ldots \text{ Prob } [X_n \leq \gamma(y)]$$

Now putting in our distribution function we have:

$$H(y) = G(\gamma(y)) \ldots G(\gamma(y)) = G(\gamma(y))^n$$

The density function, $h$ is given by

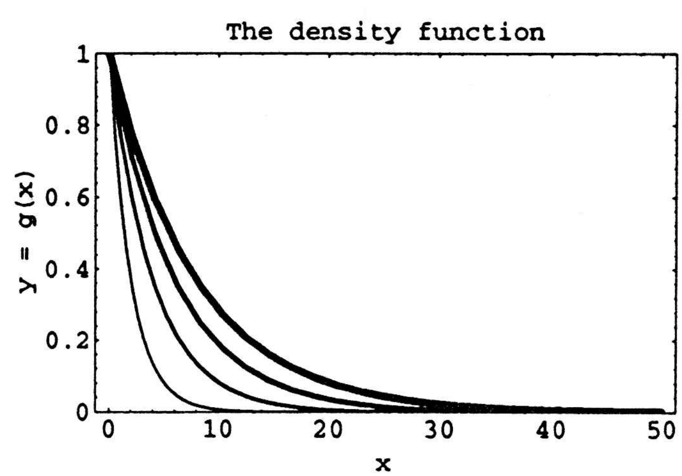
$$h(y) = nG(\gamma(y))^{n-1}g(\gamma(y))\gamma^{prime}(y).$$

So our expected selling price, $E[P] = \int_0^\infty yh(y)\,dy$.

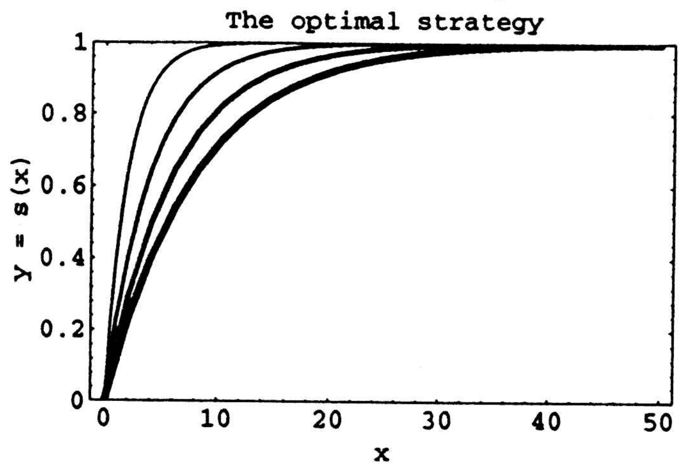
$$E[P] = \int_0^\infty ynG(y)^{n-1}g(y)\,dy$$

## 2.5 A Graphic Example

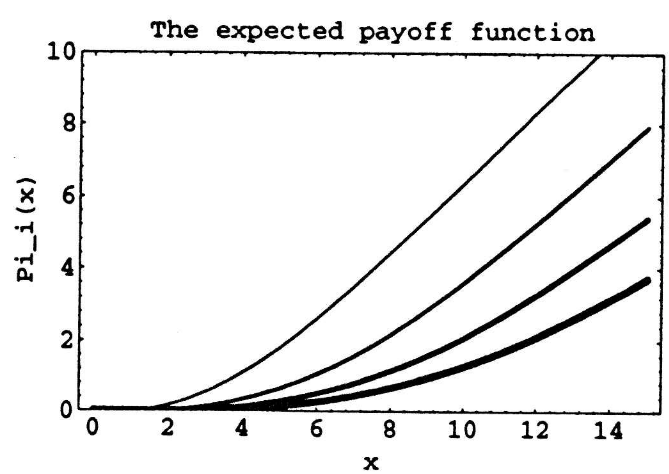
We can see more clearly what this model is demonstrating through a few graphs. Suppose we have the distribution function $G(x) = 1 - e^{-x/\theta}$, and the density function $g(x) = e^{-x/\theta}$. The following graphs show the density functions, the optimal strategy functions, and the expected payoff functions respectively. The thinest line on each graph represents $\theta = 2$, the next thickest is $\theta = 4$, then $\theta = 6$, and the thickest line on each graph is $\theta = 8$. For the functions involving $n$, $n = 4$.



As we can see from the above graph, the density function decreases less rapidly as our $\theta$ increases. This is because our negative exponent of $e$ becomes smaller and smaller.




It appears from the above graph that the slope of the strategy function decreases as our $\theta$ increases. So it seems that we would want to bid less the higher the $\theta$ is.



The graph of the expected payoff function shows that for this distribution, the smaller the $\theta$, the higher the expected payoff function will be.

# 3 Chapter 3: Modeling a Multiple Object Auction

## 3.1 Set-up

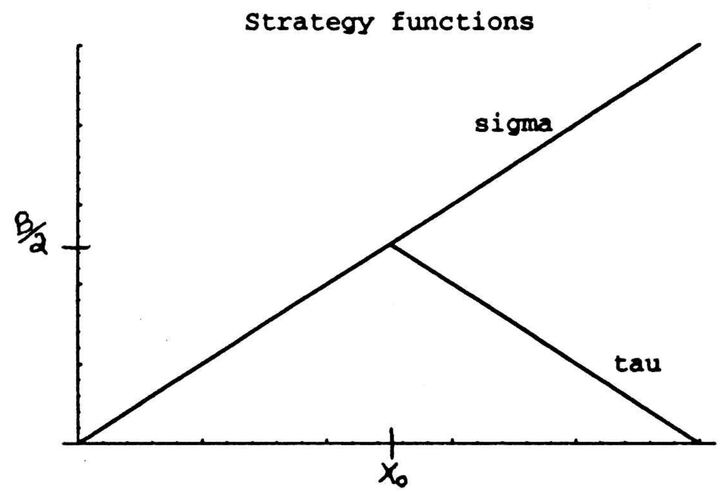
Now we will turn our attention toward modeling a multiple object auction. Once again we will begin with a set of $n$ bidders, but now we will have two identical objects up for sale. By identical, we mean that the bidders value each object equally. Let $X_1, X_2, \ldots, X_n$ be independent and identical random variables, each representing the value of each object to that bidder. So for example, $X_1$ would be player one's valuation of object one and his valuation of object two. Just as in the previous model, $G$ will be the probability distribution of the $X_i$'s and its derivative, $g$, will be the density function. For this model we will also assume that every player has the same amount of money, call this amount $B$.

We begin with the conjecture that there exists a symmetric Bayesian equilibrium, $(\sigma, \tau)$, which describes the strategy of each player. So if $X_i = x$ a player $i$ will bid $\sigma(x)$ on either object one or object two and $\tau(x)$ on the one that he did not bid $\sigma(x)$ on. Player $i$ will randomly select which strategy he will use for which object.

We will again make the assumption that $\sigma$ is an increasing, differentiable function. Also, we will assume that $\sigma(x_0) = \frac{B}{2}$. Now we make the following assumption on $\tau(x)$:

$$\tau(x) = \begin{cases} \sigma(x), & \text{if } x \leq x_0 \\ B - \sigma(x), & \text{if } x \geq x_0. \end{cases}$$

The following graphically demonstrates what these functions may look like under all of our assumptions:




## 3.2 The Expected Payoff Function

We now assume that each player $j$, when $j \neq i$, is using the Bayesian equilibrium. Our payoff function for player $i$ on object one will be the worth of object one minus player $i$'s bid times the probability that player $i$'s bid is larger than the $n-1$ other bids, denoted $p(b_1)$. So we have:

$$\pi_{i,\text{object one}}(b_1, x, \{\sigma_j\}_{j\neq i}, \{\tau_j\}_{j\neq i}) = (x - b_1)p(b_1).$$

Similarly, the payoff function for object two to player $i$ will be given by:

$$\pi_{i,\text{object two}}(b_2, x, \{\sigma_j\}_{j\neq i}, \{\tau_j\}_{j\neq i}) = (x - b_2)p(b_2).$$

Putting these two together, we have the payoff function for player $i$:

$$\pi_i(b_1, b_2, x, \{\sigma_j\}_{j\neq i}, \{\tau_j\}_{j\neq i}) = (x - b_1)p(b_1) + (x - b_2)p(b_2).$$

Because we assumed independence we may write $p(b) = \Pi_{j\neq i} q_j(b)$, where $q_j(b)$ is the probability that $b$ is larger than $j$'s bid. For the moment, let's just consider object one. Since player $j$ randomly selects whether he uses strategy $\sigma$ or strategy $\tau$ to bid on the object, each occurs with a probability of $\frac{1}{2}$. So we have

$$q_j(b) = \frac{1}{2} \text{ Prob } [\sigma(X_j) < b] + \frac{1}{2} \text{ Prob } [\tau(X_j) < b].$$

First let's consider player $i$'s bid, $b \leq \frac{B}{2}$. Then the Prob $[\tau(X_j) < b] =$ Prob $[\sigma(X_j) < b$ or $(B - \sigma(X_j)) < b]$. We can see this by looking again at the graph of our functions.

Because these two events are mutually exclusive we may write,

$$\text{Prob } [\tau(X_j) < b] = \text{ Prob } [\sigma(X_j) < b] + \text{ Prob } [B - \sigma(X_j) < b].$$

Subtracting $B$ from both sides of the second probability and then multiplying through by a negative one, we have

$$\text{Prob } [\tau(X_j) < b] = \text{ Prob } [\sigma(X_j) < b] + \text{ Prob } [\sigma(X_j) > (B - b)].$$

Now take $\sigma^{-1}$ of both sides of both probabilities.

$$\text{Prob }[\tau(X_j) < b] = \text{ Prob }[X_j < \sigma^{-1}(b)] + \text{ Prob }[X_j > \sigma^{-1}(B-b)].$$

Now let us consider player $i$'s bid $b \geq \frac{B}{2}$. Then the Prob $[\tau(X_j) < b] = 1$ we can see this clearly by looking again at the graph of our strategy functions.

So now we have

$$q_j(b) = \begin{cases} \frac{1}{2}\text{ Prob }[X_j < \sigma^{-1}(b)] \\ +\frac{1}{2}\left[\text{ Prob }[X_j < \sigma^{-1}(b)] + 1 - \text{ Prob }[X_j > \sigma^{-1}(B-b)]\right], & \text{if } b \leq \frac{B}{2} \\ \frac{1}{2}\text{ Prob }[X_j < \sigma^{-1}(b)] + \frac{1}{2}[1], & \text{if } b \leq \frac{B}{2}. \end{cases}$$

Putting our distribution function, $G$, into these equations, we can now formulate our $q_j(b)$

$$q_j(b) = \begin{cases} \frac{1}{2}G(\sigma^{-1}(b)) + \frac{1}{2}[G(\sigma^{-1}(b)) + 1 - G(\sigma^{-1}(B-b))], & \text{if } b \leq \frac{B}{2} \\ \frac{1}{2}G(\sigma^{-1}(b)) + \frac{1}{2}, & \text{if } b \geq \frac{B}{2}. \end{cases}$$

Simplifying the first equation we have

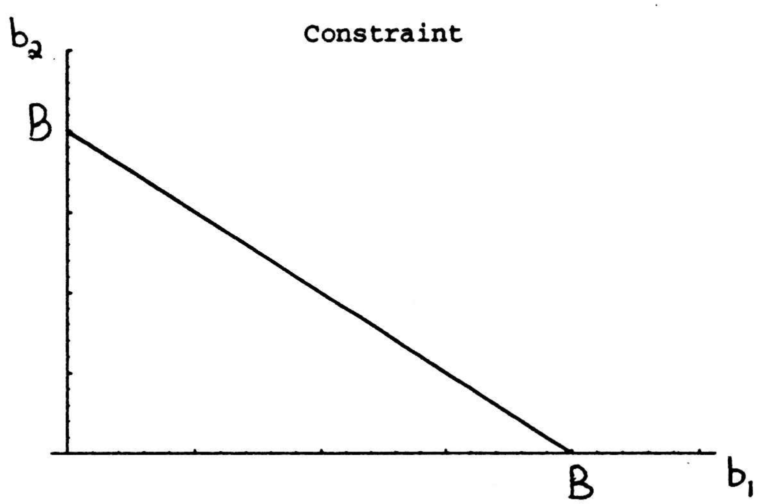
$$q_j(b) = \begin{cases} G(\sigma^{-1}(b)) + \frac{1}{2}[1 - G(\sigma^{-1}(B-b))], & \text{if } b \leq \frac{B}{2} \\ \frac{1}{2}G(\sigma^{-1}(b)) + \frac{1}{2}, & \text{if } b \geq \frac{B}{2}. \end{cases}$$

Recall that we said $p(b) = \Pi_{j \neq i} q_j(b)$, so we can now find our $p(b)$.

$$p(b) = \begin{cases} [G(\sigma^{-1}(b)) + \frac{1}{2}(1 - G(\sigma^{-1}(B - b)))]^{n-1}, & \text{if } b \leq \frac{B}{2} \\ [\frac{1}{2}G(\sigma^{-1}(b)) + \frac{1}{2}]^{n-1}, & \text{if } b \geq \frac{B}{2}. \end{cases}$$

### 3.3 Finding the Optimum Strategy

Now if $(\sigma, \tau)$ is a Bayesian equilibrium, then $\pi_i$ is maximized at $b_1 = \sigma(x)$ and $b_2 = \tau(x)$ within the constraint $0 \leq b_1 + b_2 \leq B$, graphically the constraint is:




To find the maximum of $\pi_i$ we must take the derivative and set it equal to zero. We begin by finding $p'(b)$.

$$p'(b) = \begin{cases} (n-1)\left[G(\sigma^{-1}(b)) + \frac{1}{2}[1 - G(\sigma^{-1}(B-b))]\right]^{n-2} \cdot \\ \cdot\, [g(\sigma'(b))\sigma^{-1})'(b) + \frac{1}{2}g(\sigma'(B-b))(\sigma^{-1})'(B-b)], & \text{if } b < \frac{B}{2} \\ (n-1)[\frac{1}{2}G(\sigma^{-1}(b)) + \frac{1}{2}]^{n-2}\frac{1}{2}g(\sigma'(b))(\sigma^{-1})'(b), & \text{if } b > \frac{B}{2}. \end{cases}$$

There are two cases for us to consider; the case where $x < x_0$ and the case where $x \geq x_0$.

**Case I:** If $x < x_0$, then a necessary condition that $\pi_i(b_1, b_2)$ be maximized at $b_1 = b_2 = \sigma(x)$ is that

$$\frac{\partial \pi_i}{\partial b_1} \mid_{b_1 = \sigma(x)} = 0.$$

**Case II:** If $x \geq x_0$, then a necessary condition that $\pi_i(b_1, b_2)$ be maximized at $b_1 = \sigma(x), b_2 = \tau(x)$ is that $\pi_i(b, B-b)$ be maximized at $b = \sigma(x)$, and a necessary condition for this is that

$$\frac{d}{db}\pi_i(b, B-b) \mid_{b=\sigma(x)} = 0.$$

By taking the derivative of these two equations and setting them equal to zero, I was unable to solve for $\sigma$ and $\tau$ in the general form. Therefore I was unable to solve for the optimum bidding strategies in the multiple object auction.

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[2] Freund, John E., *Mathematical Statistics*, Prentice Hall Inc., 1992.

[3] Hutchins, Robert Maynard, *The History of Herodotus*, William Benton Publisher, 1952.

[4] Kagel, John H. Alvin E. Roth, *The Handbook of Experimental Economics*, Princeton University Press, 1995.

[5] Learmount, Brian, *A History of the Auction*, Barnard Learmount, 1985.

[6] Lewyn, Mark, "What Price Air?," *Business Week*, March 14, 1994.

[7] Ross, Kenneth A., *Elementary Analysis: The Theory of Calculus*, Springer-Verlag,. 1980.

[8] U.S. Department of the Interior, *The Federal Simultaneous Oil and Gas Leasing System*, U.S. Government Printing Office, 1984.

[9] "Revenge of the Nerds," *The Economist*, July 23 to 29, 1994.

# Commands for Graphs Done by *Mathematica*

## Graph 1: The density function

```
g[x_] :=E^(-x/θ)
Plot[{E^(-x/2),E^(-x/4),E^(-x/6),E^(-x/8)}, {x, 0, 50},
Axes -> False,
Frame -> True,
FrameLabel -> {"x", "y = g(x)"},
PlotLabel -> "The density function",
PlotRange -> {0, 1},
PlotStyle -> {{Thickness[0.003]}, {Thickness[0.006]},
              {Thickness[0.009]}, {Thickness[0.012]}}]
```

## Graph 2: The optimal strategy

```
sigma[z_] := (n-1)/G[z]^(n-1)*
         Integrate[x*G[x]^(n-2)*G'[x],{x,0,z}]

G[x_] := 1-E^(-x/theta)

Plot[{1-E^(-x/2), 1-E^(-x/4), 1-E^(-x/6), 1-E^(-x/8)}, {x, 0,
Axes -> False,
Frame -> True,
FrameLabel -> {"x", "y = σ(x)"},
PlotLabel -> "The optimal strategy",
PlotRange -> {0, 1},
PlotStyle -> {{Thickness[0.003]}, {Thickness[0.006]},
              {Thickness[0.009]}, {Thickness[0.012]}}]
```

## Graph 3: The expected payoff function

```
Plot[{(1 - E^(-x/2))^3*(x - (3*

    (11/9 + (-2 + 9*E^(x/2) - 18*E^x - 3*x +

        9*E^(x/2)*x - 9*E^x*x)/(9*E^((3*x)/2))))/

  (1 - E^(-x/2))^3), (1 - E^(-x/4))^3*(x - (3*
```

```
        (22/9 + (-4 + 18*E^(x/4) - 36*E^(x/2) - 3*x +

            9*E^(x/4)*x - 9*E^(x/2)*x)/(9*E^((3*x)/4))))/

      (1 - E^(-x/4))^3), (1 - E^(-x/6))^3*(x - (3*

        (11/3 + (-2 + 9*E^(x/6) - 18*E^(x/3) - x +

            3*E^(x/6)*x - 3*E^(x/3)*x)/(3*E^(x/2))))/

      (1 - E^(-x/6))^3), (1 - E^(-x/8))^3*(x - (3*

        (44/9 + (-8 - x)/E^(x/8) + (4 + x)/E^(x/4) -

   (8 + 3*x)/(9*E^((3*x)/8))))/(1 - E^(-x/8))^3)}, {x, 0, 15

          Axes -> False,
          Frame -> True,
          FrameLabel -> {"x", "Pi_i(x)"},
          PlotLabel -> "The expected payoff function",
          PlotRange -> {0, 10},
          PlotStyle -> {{Thickness[0.003]}, {Thickness[0.006
                        {Thickness[0.009]}, {Thickness[0.01:
```

# Graph 4: The strategy functions

```
sigma[x_] := x
tau[x_] := x      /; x<=5
tau[x_] := 10-x   /; x>=5
Plot[{sigma[x], tau[x]}, {x, 0, 10},
    Axes -> True,
    FrameLabel -> {"X_i", "Bid"},
    PlotLabel -> "Strategy functions",
    PlotRange -> {0, 10}]
```

```
Show[g1, Graphics[Text["sigma", {7, 8}]],
     Graphics[Text["tau", {9, 2}]]]
```

# Graph 5: The constraint

```
f[x_] := 8 -x
Plot[{f[x]}, {x, 0, 10},
    Axes -> True,
    FrameLabel -> {"b_1", "b_2"},
    PlotLabel -> "Constraint",
    PlotRange -> {0, 10}]
```