13 Limits, Continuity, and the Definition of Derivative

Chapter 2 Focus on Theory

An Example

Suppose the distance (in light-years) of the Starship Enterprise from the Earth at t years after its mission started is given by $f(t) = 8\sin\left(\frac{\pi}{5}t\right) + \sin(7\pi t)$.

1. How far from the Earth is the Starship Enterprise 2.1 years after its mission started?

```
f[t_] := 8 Sin[π/5 t] + Sin[7 π t]
```

```
f[2.1]
```

```
8.55768
```

About 8.56 light-years.

2. What was the average velocity (with respect to the Earth) of the Starship Enterprise during the first 2.1 years of its mission?

```
(f[2.1] - f[0]) / (2.1 - 0)
```

```
4.07509
```

About 4.08 light-years per year (warp 4).

3. What was the instantaneous velocity (with respect to the Earth) of the Starship Enterprise 2.1 years after its mission started?

```
f[2.10001] - f[2.1]
-------------------
  2.10001 - 2.1
```

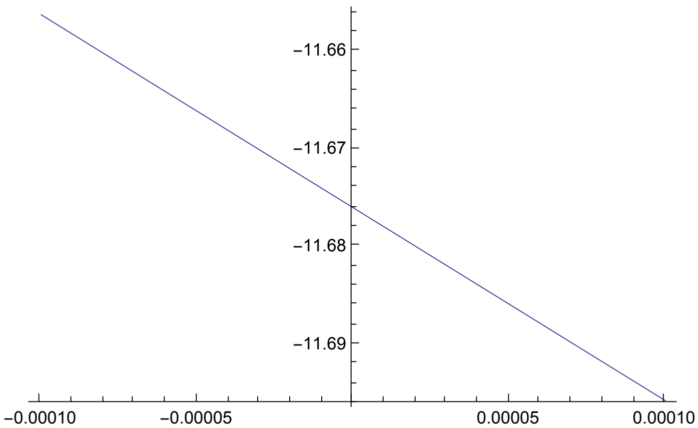
```
-11.678
```

About warp 11.68.

```
Table[(f[2.1 + Δt] - f[2.1])/Δt, {Δt, {.0001, .00001, .000001}}]
```

```
{-11.6957, -11.678, -11.6762}
```

```
Plot[(f[2.1 + Δt] - f[2.1])/Δt, {Δt, -.0001, .0001}]
```



Limits

Definition. If the values of $g(x)$ get closer and closer to a finite number as x gets closer and closer to a, we call that finite number the *limit* of $g(x)$ as x approaches a, and denote it by $\lim_{x \to a} g(x)$.

1. What is the limit of $g(x) = e^{-\tan(\pi x)}$ as x approaches 0.3?

```
g[x_] := e^(-Tan[π x])
Table[g[x], {x, {0.301, 0.3001, 0.30001}}]
```

```
{0.250195, 0.252261, 0.252467}
```

About 0.253.

2. What is the limit of $g(x) = e^{-\tan(\pi x)}$ as x approaches 0.5?

```
g[x_] := e^(-Tan[π x])
Table[g[x], {x, {0.501, 0.5001, 0.50001}}]
```

$\{1.73689 \times 10^{138}, 2.524791790952 \times 10^{1382}, 1.05367341316 \times 10^{13824}\}$

Does not exist.

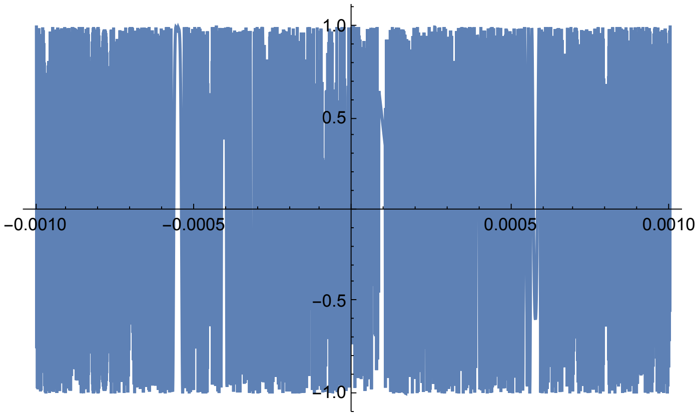
3. What is the limit of $h(x) = \cos\left(\frac{\pi}{x}\right)$ as x approaches 0?

```
In[11]:= h[x_] := Cos[π / x]
Table[h[x], {x, {0.001, 0.0001, 0.00001, 0.0000010133}}]
```

```
Out[12]= {1., 1., 1., -0.212751}
```

```
Plot[h[x], {x, -0.001, 0.001}]
```

Out[4]=



4. What is the limit of $\frac{f[2.1+\Delta t]-f[2.1]}{\Delta t}$ as Δt approaches 0?

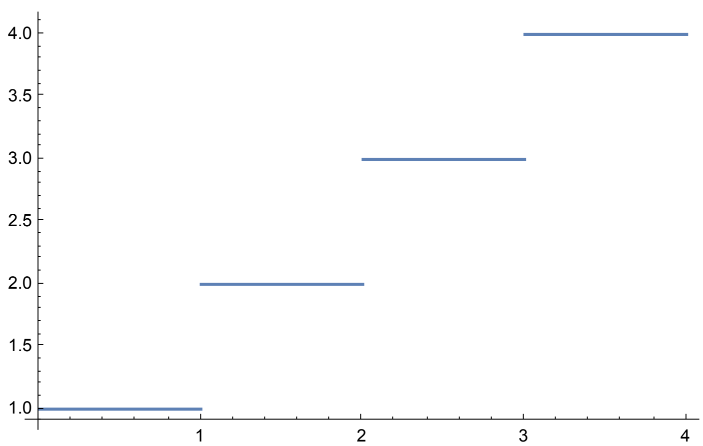
About −11.676.

$$\text{Limit}\left[\frac{f[2.1+\Delta t]-f[2.1]}{\Delta t}, \Delta t \to 0\right]$$

-11.676

5. What is the limit of the ceiling function as *x* approaches 2?

```
Plot[Ceiling[x], {x, 0, 4}]
Limit[Ceiling[x], x → 2, Direction → -1]
Limit[Ceiling[x], x → 2, Direction → 1]
Limit[Ceiling[x], x → 2]
```



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3

Continuity

Definition. A function $g(x)$ is *continuous* at $x = a$ if $\lim_{x\to a} g(x) = g(a)$.

Remark. Implicit in this statement are three criteria: (1) the function g(x) must be defined at a, (2) the limit of f(x) as x approaches a must exist, and (3) the two must be equal.

1. Is $g(x) = e^{-\tan(\pi x)}$ continuous at $x = 0.3$?

```
Limit[g[x], x → 0.3]
g[0.3]
```

0.25249

0.25249

2. Is $g(x) = e^{-\tan(\pi x)}$ continuous at $x = 0.5$?

```
Limit[g[x], x → 0.5]
g[0.5]
```

∞

General::unfl : Underflow occurred in computation. ≫

Underflow[]

3. Is $h(\Delta t) = \frac{f(2.1+\Delta t) - f(2.1)}{\Delta t}$ continuous at $\Delta t = 0$?

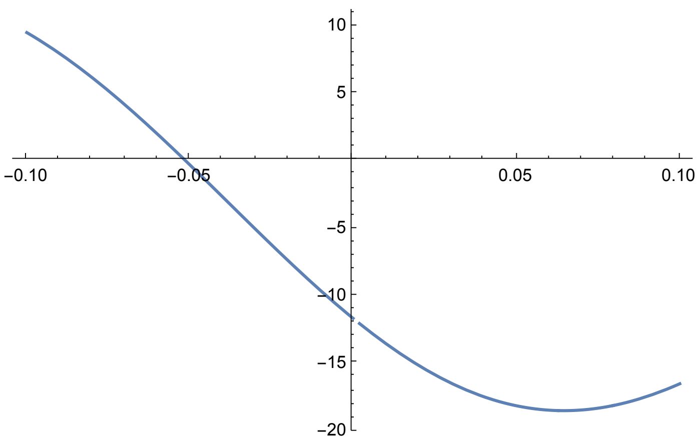
```
Limit[(f[2.1 + Δt] - f[2.1])/Δt, Δt → 0]
(f[2.1 + Δt] - f[2.1])/Δt /. Δt → 0
```

-11.676

Power::infy : Infinite expression $\frac{1}{0}$ encountered. ≫

ComplexInfinity

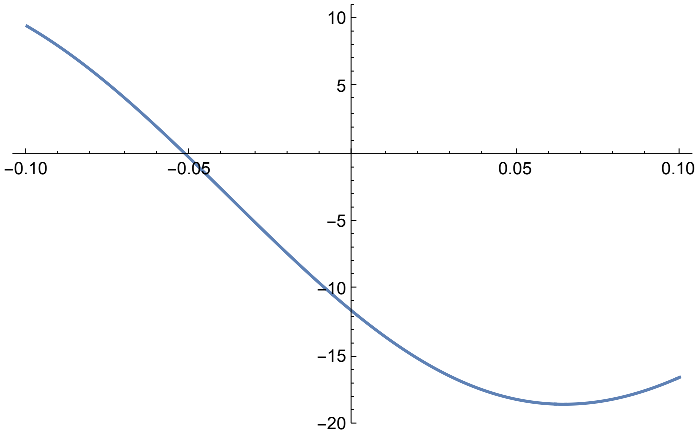
```
Plot[(f[2.1 + Δt] - f[2.1])/Δt, {Δt, -.1, .1}, Exclusions → {Δt == 0}]
```



We can define a continuous extension of $h(\Delta t) = \frac{f(2.1+\Delta t)-f(2.1)}{\Delta t}$:

```
h[Δt_] := Piecewise[{{(f[2.1 + Δt] - f[2.1])/Δt, Δt ≠ 0}, {-11.67602109737054`, Δt == 0}}]
```

```
Plot[h[x], {x, -.1, .1}, PlotPoints → 200]
```



4. Is the ceiling function continuous at $x = 3$?

No, since the limit does not exist.

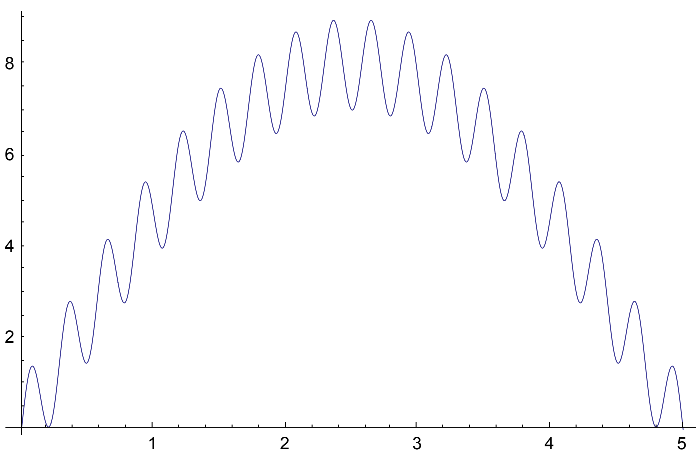
Derivatives Analytically

1. If $f(x) = x^2 - 4x + 7$, find $f'(3)$.

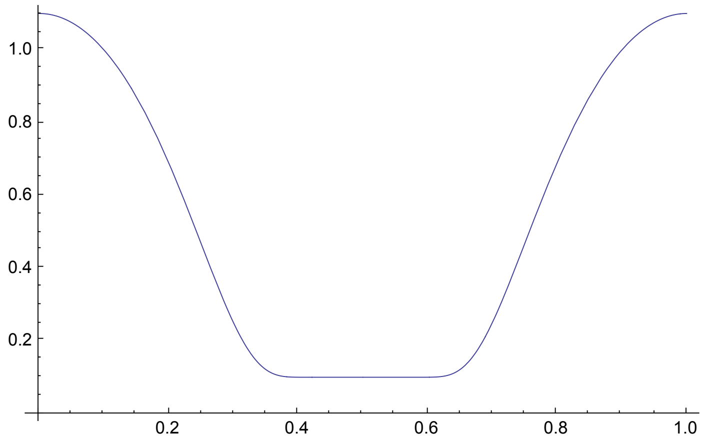
2. If $f(x) = x^2 - 4x + 7$, find $f'(x)$.

Sandbox

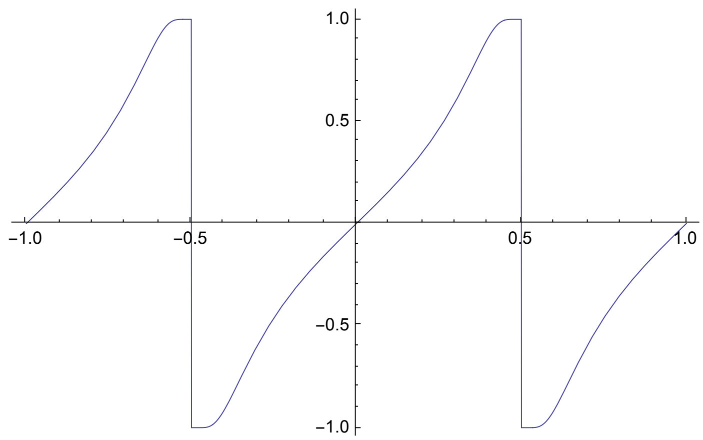
```
Plot[f[t], {t, 0, 5}]
```



```
f[x_] := e^(-Tan[π x]^2)
Plot[f[x] + .1, {x, 0, 1}, Exclusions → 1 / 2, ExclusionsStyle → None]
```



```
f[x_] := (1 - e^(-Tan[π x])) / (1 + e^(-Tan[π x]))
Plot[f[x], {x, -1, 1}]
```



```
f[.5]
```

General::unfl : Underflow occurred in computation. ≫

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```
Plot[(1 - e^-x) / (1 + e^-x), {x, -10, 10}]
```