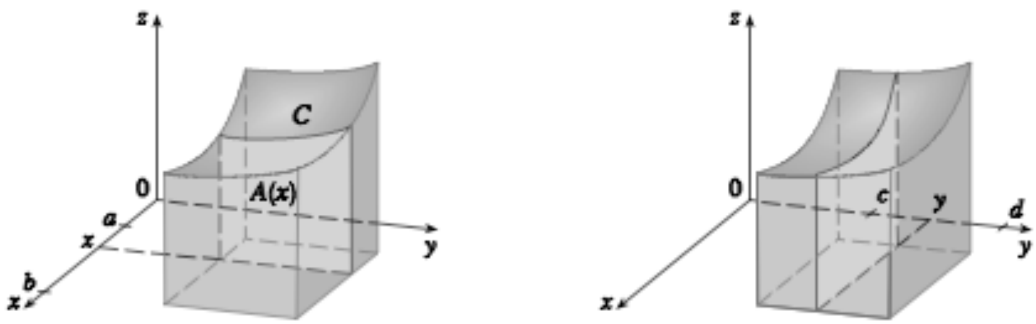
Math 213 Calculus III

Reading the Text

Read Section 12.1-12.3 and answer the following questions

1. Compute $\sum_{i=1}^{2}\sum_{j=1}^{3} 2^i 3^j$
2. If we partition [a, b] into m subintervals of equal length and [c, d] into n subintervals of equal length, what is the value of ΔA for any subrectangle R_{ij} ?
3. Consider Figures 1 and 2 in Section 12.2 in the text. Why is $\int_a^b A(x)\,dx = \int_c^d A(y)\,dy$?



4. Compute $\int_0^3 \int_3^4 x^2 y\,dy\,dx$
5. Is it true that $\int_0^1 \int_x^1 f(x,y)\,dy\,dx = \int_0^1 \int_y^1 f(x,y)\,dx\,dy$?

Class 09: Optimization (Unbounded Region)

For the following functions
(a) Find critical points.
(b) Find max/min/saddle points if they exist

$f(x,y) = 4xy - x^4 - y^4$

$f(x,y) = x^2 - 2y^2 - 6x + 8y + 3$

$f(x,y) = x^2 + 2y^2 - 6x + 8y - 1$

Class 09: Optimization (Bounded Regions)

Suppose we want to find the maximum and minimum value of $f(x,y) = 4x^2 - 3y^2 + 2xy$ on the square $0 \le x \le 1$, $0 \le y \le 1$.

Does *f* have a critical point in the interior of the square?

Find critical values for *f* on the boundary by examining how *f* behaves on each of the line segments which make up the boundary.

On $x = 0$

On $x = 1$

On $y = 0$

On $y = 1$

What are the absolute maxima and minima of *f* over the square?

Class 09: Lagrange Multipliers Notes

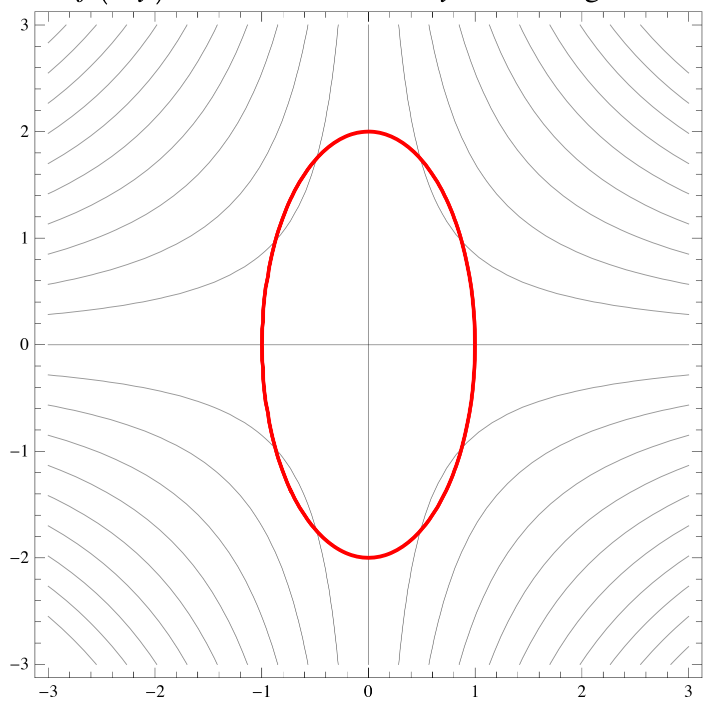
Introduction

Solving optimization problems using the technique of Lagrange multipliers is often time–consuming. These problems involve setting up the problem, performing differentiation to obtain a system of equations, solving the system for potential extrema, and examining points for extrema. Often the system of equations of non-linear, and even solving a linear system can be time consuming, especially if we have many variables and more than one constraint.

Functions of 2 Variables with One Constraint

Example

Suppose we want to find the minimum value of $f(x,y) = xy$ subject to $4x^2 + y^2 = 4$. When we look at the level curves of $f(x,y)$ and the curve $4x^2 + y^2 = 4$, we get the following graph:



We want to find x and y values, such that the level curve, $c = xy$, is tangent to the curve $4x^2 + y^2 = 4$. If we let $f(x,y) = xy$ and we let $g(x,y) = 4x^2 + y^2$, then we want ∇f to be parallel to ∇g. Or, we want $\nabla f = \lambda \nabla g$. Since $\nabla f = [y,x]$ and $\nabla g = [8x,2y]$, setting these two gradients equal gives us the equations:

$$y = \lambda(8x)$$
$$x = \lambda(4y)$$

Using these two equations along with the constraint equation $4x^2 + y^2 = 4$, we have a system of 3 equations and 3 unknowns (x,y,λ).

Class 09: Lagrange Multipliers Notes

We can solve for this system, and we get the following solutions:

$$(x,y)=\left(-\frac{1}{\sqrt{2}},\sqrt{2}\right),\quad f\left(-\frac{1}{\sqrt{2}},\sqrt{2}\right)=-1$$

$$(x,y)=\left(\frac{1}{\sqrt{2}},\sqrt{2}\right),\quad f\left(\frac{1}{\sqrt{2}},\sqrt{2}\right)=1$$

$$(x,y)=\left(-\frac{1}{\sqrt{2}},-\sqrt{2}\right),\quad f\left(-\frac{1}{\sqrt{2}},-\sqrt{2}\right)=1$$

$$(x,y)=\left(\frac{1}{\sqrt{2}},-\sqrt{2}\right),\quad f\left(\frac{1}{\sqrt{2}},-\sqrt{2}\right)=-1$$

Therefore, the minimum value of $f(x,y)$ subject to the constraint $4x^2+y^2=4$ is –1 and occurs when $(x,y)=\left(-\frac{1}{\sqrt{2}},\sqrt{2}\right)$, $(x,y)=\left(\frac{1}{\sqrt{2}},-\sqrt{2}\right)$. The maximum value of $f(x,y)$ subject to the constraint $4x^2+y^2=4$ is +1 and occurs when $(x,y)=\left(\frac{1}{\sqrt{2}},\sqrt{2}\right)$, $(x,y)=\left(-\frac{1}{\sqrt{2}},-\sqrt{2}\right)$.

Functions of 3 Variables with One Constraint

This same technique can be used for functions of three variables. We will then get a system of 4 equations and 4 unknowns.

Functions of 3 or More Variables with Two Constraints

Example:

We can also minimize (or maximize) functions of more than three variables.
With a little modification, we can minimize (or maximize) functions with more than one constraint.

Suppose we want to find the minimum value of the function $f(x,y,z,t)=x^2+2y^2+z^2+t^2$ subject to the constraints $\begin{cases} x+3y-z+t=2 \\ 2x-y+z+2t=4 \end{cases}$.

We define

$$f:=x^2+2y^2+z^2+t^2$$
$$g:=x+3y-z+t-2$$
$$h:=2x-y+z+2t-4$$

Now, we want to find the solution to the problem $\nabla f=\lambda\nabla g+\mu\nabla h$. Taking the gradients and setting the respective components (in 4 dimensions now) equal gives us 4 equations. Combining that with the two constraint equations gives us a system of 6 equations in 6 unknowns (x,y,z,t,λ,μ).

$$\begin{aligned}
2x &= \lambda + 2\mu \\
4y &= 3\lambda - \mu \\
2z &= -\lambda + \mu \\
2t &= \lambda + 2\mu \\
x + 3y - z + t &= 2 \\
2x - y + z + 2t &= 4
\end{aligned}$$

We can solve the system and we should get the solution:

$$x = \frac{67}{69}, \quad y = \frac{2}{23}, \quad z = \frac{14}{69}, \quad t = \frac{67}{69}, \quad \lambda = -\frac{26}{69}, \quad \mu = \frac{18}{23}$$

If we now evaluate *f* at this point, we get a value of $\frac{134}{69}$. This is the minimum value of *f* subject to the given constraints.

Functions of 3 or More Variables with More than Two Constraints

Example:

We can also minimize (or maximize) functions when we have more than 2 constraints.. With a little modification, we can minimize (or maximize) functions with more than one constraint.

Suppose we want to find the minimum of the function $f(x,y,z,t) = 2x^2 + y^2 + z^2 + 2t^2$

subject to the constraints $\begin{cases} x + y - z - t = 1 \\ 2x + y - z + 2t = 2 \\ x - y + z - t = 4 \end{cases}$

We define

$$\begin{aligned}
f &:= 2x^2 + y^2 + z^2 + 2t^2 \\
g &:= x + y - z - t - 1 \\
h &:= 2x + y - z + 2t - 2 \\
k &:= x - y + z - t - 4
\end{aligned}$$

Now, we want to find the solution to the problem $\nabla f = \lambda \nabla g + \mu \nabla h + \sigma \nabla k$.

Nonlinear Problems

The examples so far have given us systems of linear equations. What if we don't have a system of linear equations. We can still use the technique of Lagrange multipliers

Class 09: Lagrange Multiplier Problems

For each of the following, find the minimum or the maximum of the function *f* and also give the *x*, *y*, (*z*, *t*) values at which the minimum or maximum occurs.

1. Find the minimum value of $f(x,y) = x^2 - 8x + y^2 - 12y + 48$ subject to the constraint $x + y = 8$.

   minimum value of *f*:

   (x,y) values at which the minimum occurs:

2. Find the minimum value of $f(x,y,z) = 2x^2 + y^2 + 3z^2$ subject to the constraint $2x - 3y - 4z = 49$.

   minimum value of *f*:

   (x,y,z) values at which the minimum occurs:

3. Find the maximum value of $f(x,y,z) = xy + yz$ subject to the constraints $\begin{cases} x + 2y = 6 \\ x - 3z = 0 \end{cases}$.

   maximum value of *f*:

   (x,y,z) values at which the maximum occurs:

4. Find the maximum value of the function $f(x,y,z) = xyz$ subject to the constraint $x + y + z - 6 = 0$.

   maximum value of *f*:

   (x,y,z) values at which the maximum occurs: