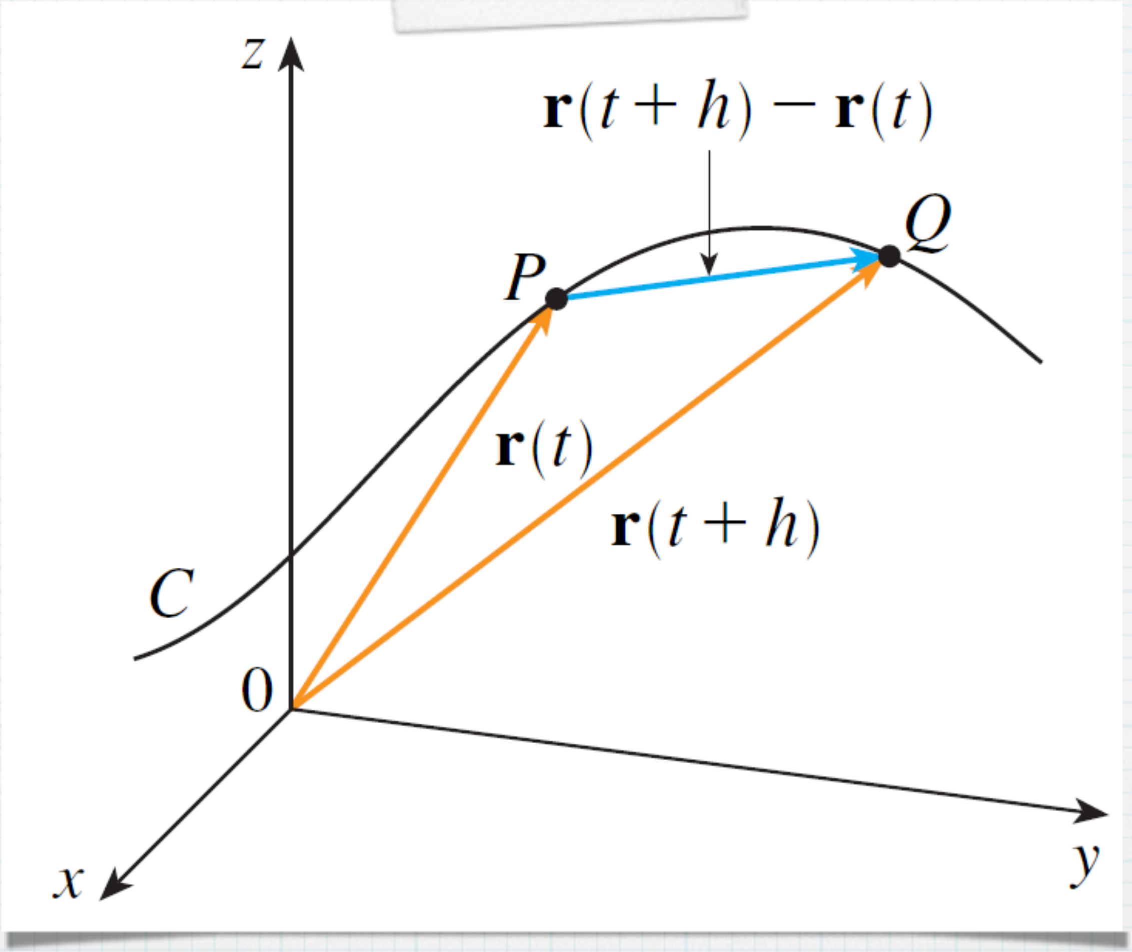
# Section 10.2

## Derivatives and Integrals of Vector Functions

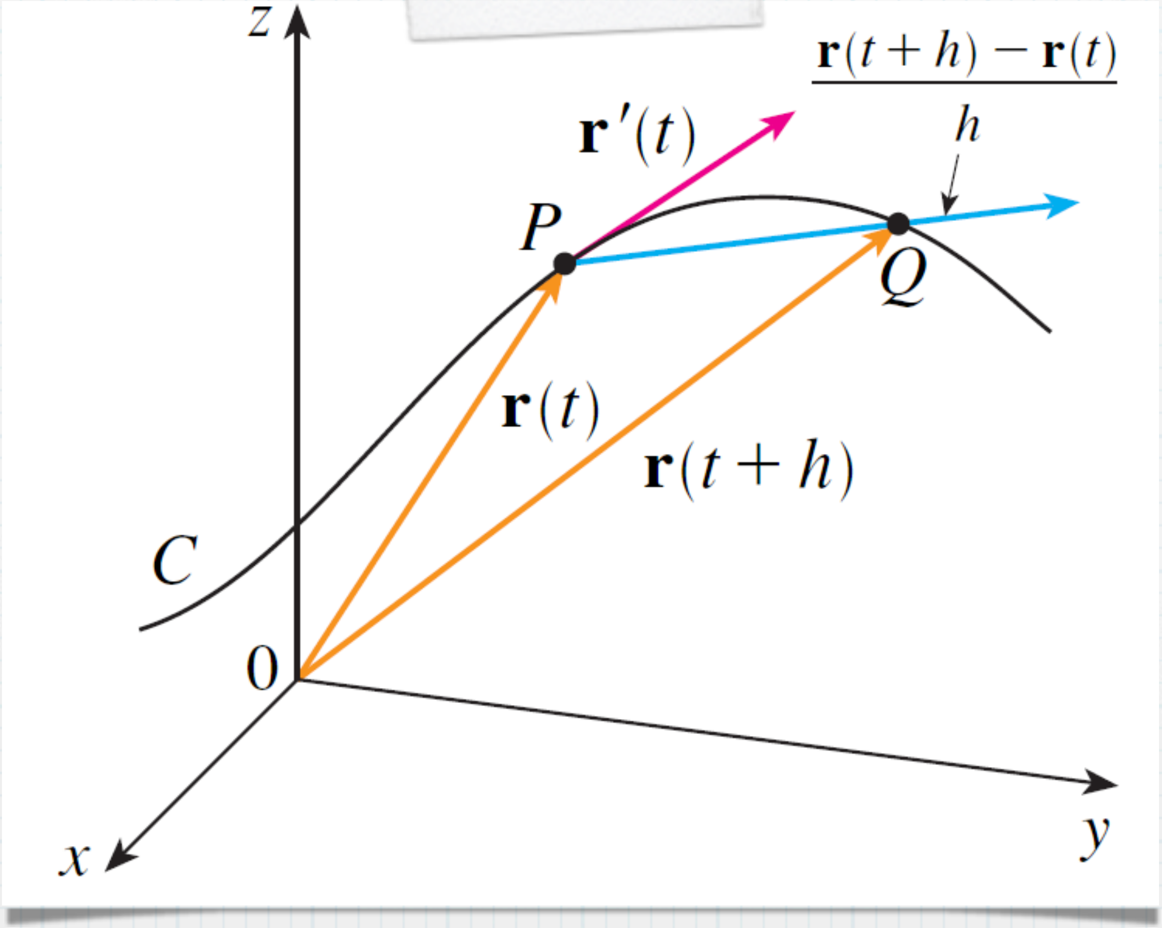
# Derivatives

$$\frac{d\mathbf{r}}{dt} = \mathbf{r}'(t) = \lim_{h \to 0} \frac{\mathbf{r}(t + h) - \mathbf{r}(t)}{h}$$




# Geometry

# Tangent Vector



# Theorem

$\boxed{2}$ **Theorem** If $\mathbf{r}(t) = \langle f(t), g(t), h(t) \rangle = f(t)\,\mathbf{i} + g(t)\,\mathbf{j} + h(t)\,\mathbf{k}$, where $f$, $g$, and $h$ are differentiable functions, then

$$\mathbf{r}'(t) = \langle f'(t), g'(t), h'(t) \rangle = f'(t)\,\mathbf{i} + g'(t)\,\mathbf{j} + h'(t)\,\mathbf{k}$$

# Differentiation Rules

**3 Theorem** Suppose $\mathbf{u}$ and $\mathbf{v}$ are differentiable vector functions, $c$ is a scalar, and $f$ is a real-valued function. Then

1. $$\frac{d}{dt}[\mathbf{u}(t) + \mathbf{v}(t)] = \mathbf{u}'(t) + \mathbf{v}'(t)$$

2. $$\frac{d}{dt}[c\mathbf{u}(t)] = c\mathbf{u}'(t)$$

3. $$\frac{d}{dt}[f(t)\mathbf{u}(t)] = f'(t)\mathbf{u}(t) + f(t)\mathbf{u}'(t)$$

4. $$\frac{d}{dt}[\mathbf{u}(t) \cdot \mathbf{v}(t)] = \mathbf{u}'(t) \cdot \mathbf{v}(t) + \mathbf{u}(t) \cdot \mathbf{v}'(t)$$

5. $$\frac{d}{dt}[\mathbf{u}(t) \times \mathbf{v}(t)] = \mathbf{u}'(t) \times \mathbf{v}(t) + \mathbf{u}(t) \times \mathbf{v}'(t)$$

6. $$\frac{d}{dt}[\mathbf{u}(f(t))] = f'(t)\mathbf{u}'(f(t))$$ (Chain Rule)

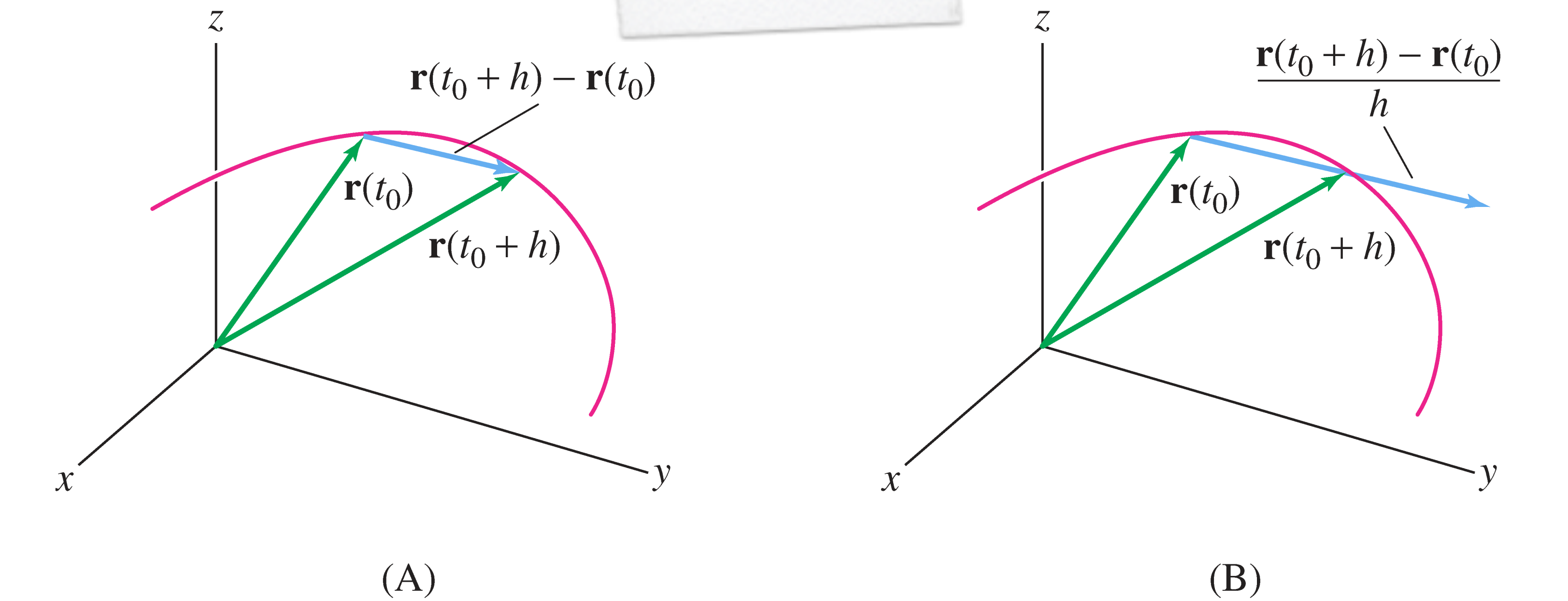
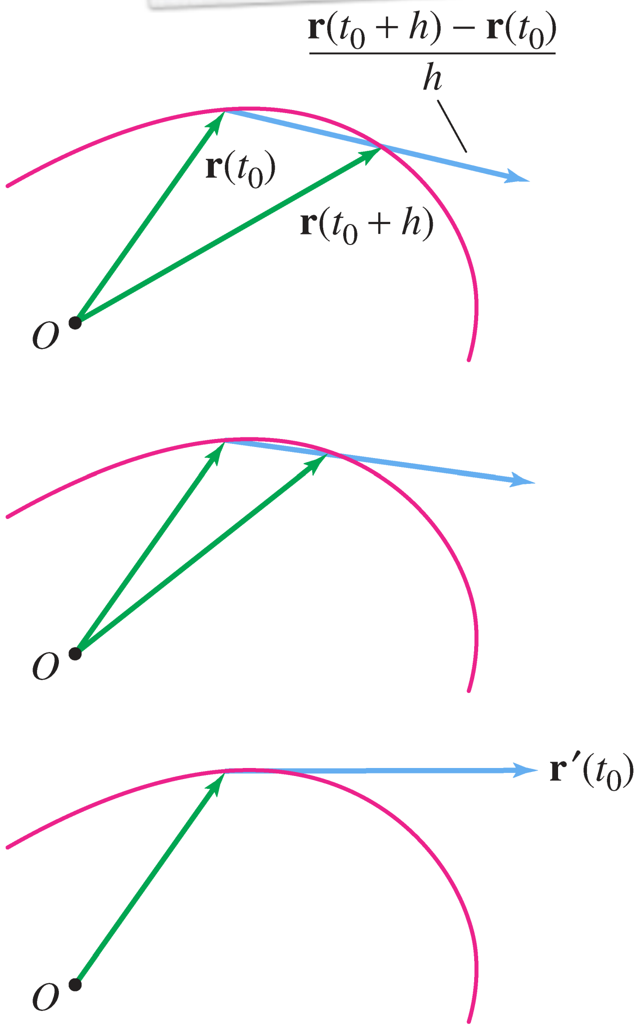



**FIGURE 2** The difference quotient points in the direction of $\Delta\mathbf{r} = \mathbf{r}(t_0 + h) - \mathbf{r}(t_0)$.

# Tangent Vector




Vector pointing in direction tangent to the curve at the point $t_0$

**FIGURE 3** The difference quotient converges to a vector $\mathbf{r}'(t)$, tangent to the curve $\mathbf{r}(t)$.

Tangent line at $\mathbf{r}(t_0)$: $\quad \mathbf{L}(t) = \mathbf{r}(t_0) + t\mathbf{r}'(t_0)$

# Unit Tangent

$$T(t) = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|}$$

* The Unit Tangent has constant length 1.
* The only quantity that changes over time is

# Theorem

* If a vector function has constant length, then its derivative is perpendicular to the vector
* Example: r(t)=<cos(t),sin(t)>

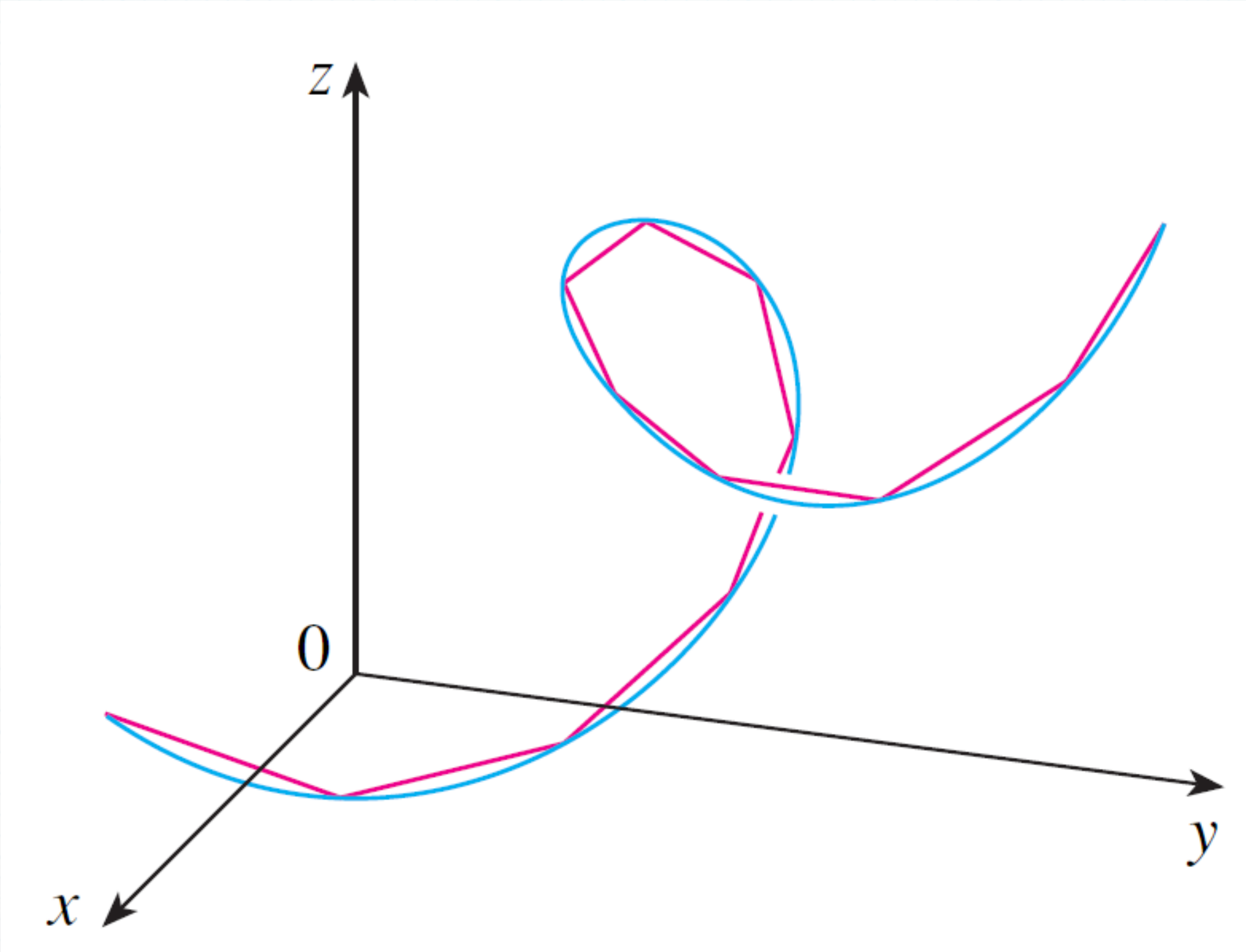
# Proof

$$0 = \frac{d}{dt}[\mathbf{r}(t) \cdot \mathbf{r}(t)] = \mathbf{r}'(t) \cdot \mathbf{r}(t) + \mathbf{r}(t) \cdot \mathbf{r}'(t) = 2\mathbf{r}'(t) \cdot \mathbf{r}(t)$$

# Section 10.3

## Arc Length and Curvature

# Arc Length



# Arc Length

$$\mathbf{r}(t) = \langle f(t), g(t), h(t) \rangle, a \le t \le b$$

$$L = \int_a^b \sqrt{[f'(t)]^2 + [g'(t)]^2 + [h'(t)]^2}\, dt$$

$$= \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\, dt$$

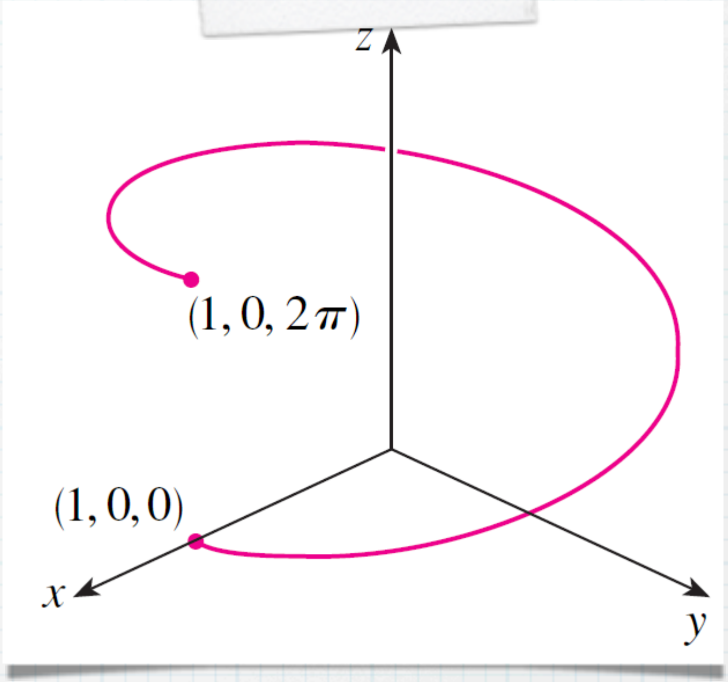
$$L = \int_a^b |\mathbf{r}'(t)|\, dt$$

# Example

- Find the length of the arc of the circular helix with vector equation

  r(t)=<cos(t),sin(t),t>

  from the point (1, 0, 0) to (1, 0, 2π).




$$\boxed{|\mathbf{r}'(t)| = \sqrt{(-\sin t)^2 + \cos^2 t + 1} = \sqrt{2}}$$

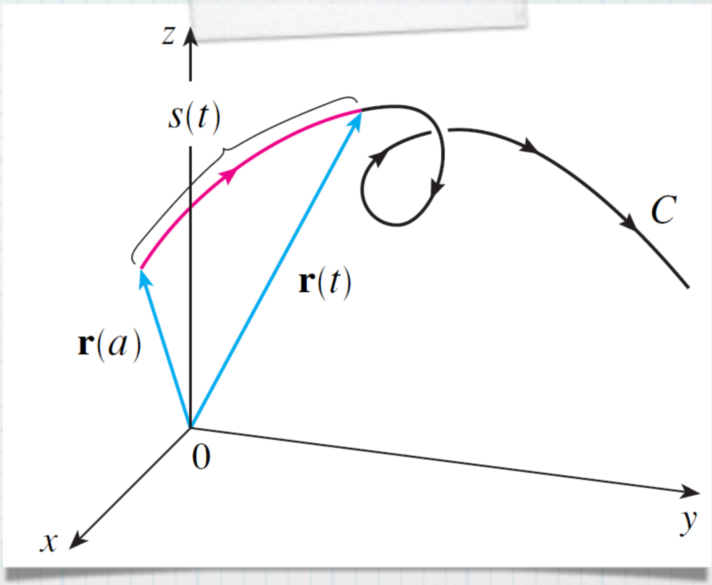
$$\boxed{L = \int_0^{2\pi} |\mathbf{r}'(t)|\, dt = \int_0^{2\pi} \sqrt{2}\, dt = 2\sqrt{2}\,\pi}$$

# Arc Length Function

$$\mathbf{r}(t) = \langle f(t), g(t), h(t) \rangle, a \le t \le b$$

$$s(t) = \int_a^t |\mathbf{r}'(u)|\, du = \int_a^t \sqrt{\left(\frac{dx}{du}\right)^2 + \left(\frac{dy}{du}\right)^2 + \left(\frac{dz}{du}\right)^2}\, du$$

$$\frac{ds}{dt} = |\mathbf{r}'(t)|$$



# Parametrizations

* A single curve can be represented by more than one vector function.
* Example: The twisted cubic

$$\mathbf{r}_1(t) = \langle t, t^2, t^3 \rangle \qquad 1 \leqslant t \leqslant 2$$

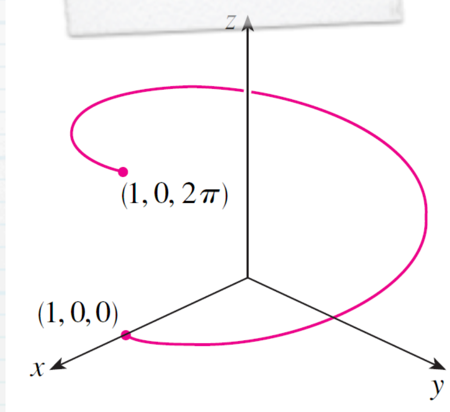
$$\mathbf{r}_2(u) = \langle e^u, e^{2u}, e^{3u} \rangle \qquad 0 \leqslant u \leqslant \ln 2$$

# Re-parametrization

- It is often useful to parametrize a curve with respect to arc length because arc length comes from the curve itself and does not depend on any one coordinate system.
- Example: $r(t)=\langle\cos(t),\sin(t),t\rangle$
  Reparametrize with respect to arc length, beginning at (1,0,0) in direction of increasing t

# Re-parametrize

- Vector function: r(t)=<cos(t),sin(t),t>
- The vector <1,0,0> corresponds to t=0




$$|\mathbf{r}'(t)| = \sqrt{(-\sin t)^2 + \cos^2 t + 1} = \sqrt{2}$$

$$s = s(t) = \int_0^t |\mathbf{r}'(u)|\,du = \int_0^t \sqrt{2}\,du = \sqrt{2}\,t$$

$$\mathbf{r}(t(s)) = \cos\!\left(s/\sqrt{2}\right)\mathbf{i} + \sin\!\left(s/\sqrt{2}\right)\mathbf{j} + \left(s/\sqrt{2}\right)\mathbf{k}$$

# Tangent Vectors and Arc Length Parametrization

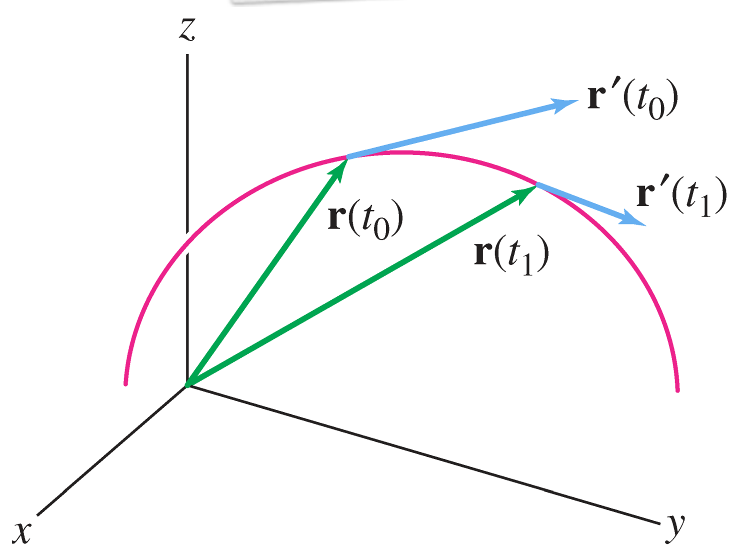
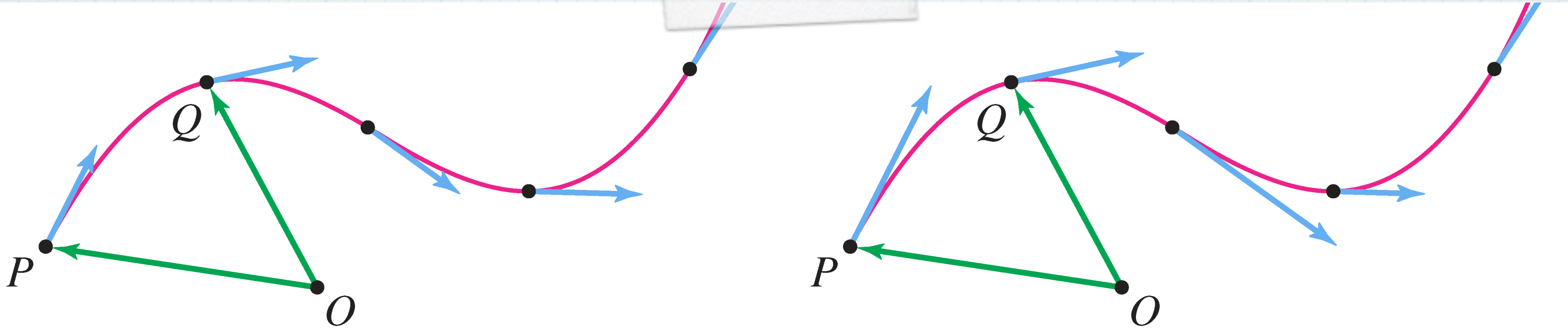



**FIGURE 2** The particle is moving faster at $t_0$ than $t_1$ since the velocity vector is longer at $t_0$.

# Tangent Vectors and Arc Length Parametrization

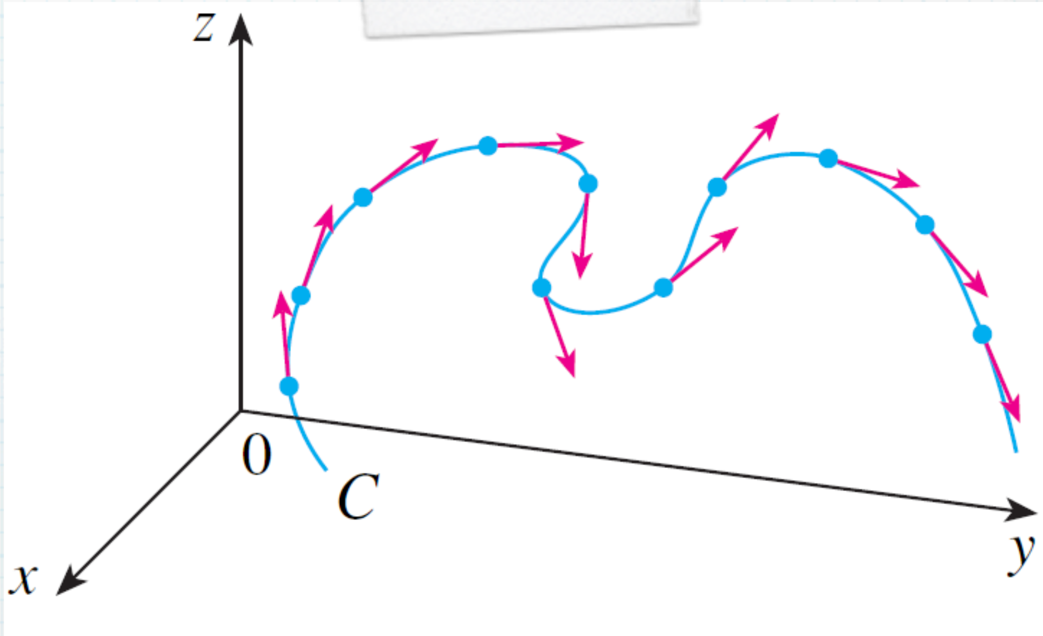



(A) An arc length parametrization (all tangent vectors have length 1)

(B) Not an arc length parametrization (tangent vectors' lengths vary)

# Curvature

- The curvature of a curve, C, at a given point is a measure of how quickly the curve changes direction at that point.
- Look at how the unit tangent changes direction.
- T(t) changes direction...
  - slowly when C is fairly straight, but
  - quickly when C bends or twists more sharply:




$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|}$$

# Unit Tangent and Curvature

$$\text{Unit tangent vector} = \mathbf{T}(t) = \frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|}$$

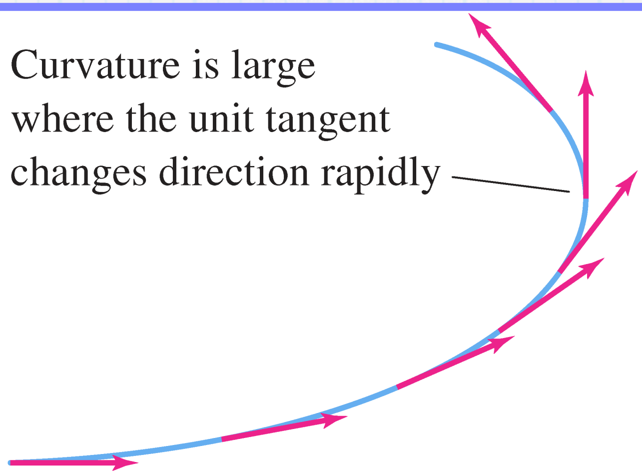



**FIGURE 4** The unit tangent vector varies in direction but not length.

# Curvature

**8 Definition** The **curvature** of a curve is

$$\kappa = \left| \frac{d\mathbf{T}}{ds} \right|$$

where $\mathbf{T}$ is the unit tangent vector.

$$\kappa(t) = \frac{|\mathbf{T}'(t)|}{|\mathbf{r}'(t)|}$$

# Remarks

* Note
  * <u>small</u> circles have <u>large</u> curvature and
  * <u>large</u> circles have <u>small</u> curvature,

in accordance with our intuition.

* The curvature of a straight line is always 0 because the tangent vector is constant.

# Theorem

$\boxed{10}$ **Theorem** The curvature of the curve given by the vector function $\mathbf{r}$ is

$$\kappa(t) = \frac{|\mathbf{r}'(t) \times \mathbf{r}''(t)|}{|\mathbf{r}'(t)|^3}$$

# Example

$$\mathbf{r}(t) = \langle t, t^2, t^3 \rangle$$

$$\mathbf{r}'(t) = \langle 1, 2t, 3t^2 \rangle \qquad \mathbf{r}''(t) = \langle 0, 2, 6t \rangle$$

$$|\mathbf{r}'(t)| = \sqrt{1 + 4t^2 + 9t^4}$$

$$\mathbf{r}'(t) \times \mathbf{r}''(t) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2t & 3t^2 \\ 0 & 2 & 6t \end{vmatrix} = 6t^2\,\mathbf{i} - 6t\,\mathbf{j} + 2\,\mathbf{k}$$

$$|\mathbf{r}'(t) \times \mathbf{r}''(t)| = \sqrt{36t^4 + 36t^2 + 4} = 2\sqrt{9t^4 + 9t^2 + 1}$$

$$\kappa(t) = \frac{|\mathbf{r}'(t) \times \mathbf{r}''(t)|}{|\mathbf{r}'(t)|^3} = \frac{2\sqrt{1 + 9t^2 + 9t^4}}{(1 + 4t^2 + 9t^4)^{3/2}}$$

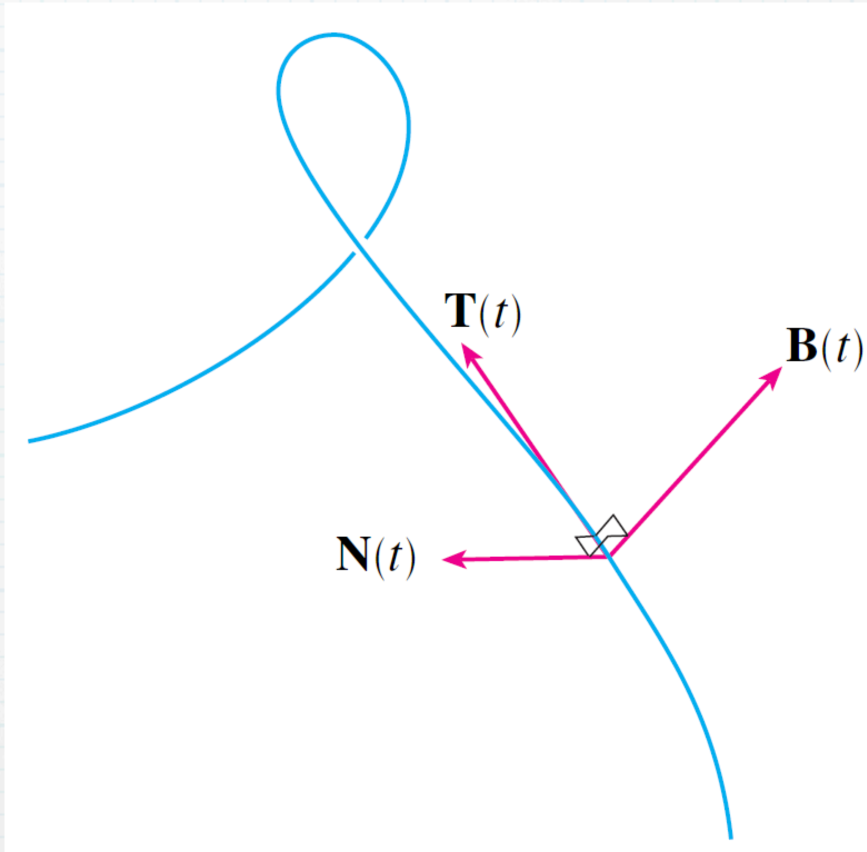
# Unit Tangent, Unit Normal and Binormal

$$\mathbf{r}(t)$$

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{\left|\mathbf{r}'(t)\right|}$$

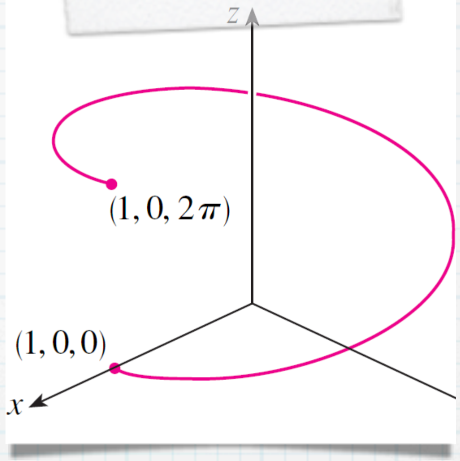
$$\mathbf{N}(t) = \frac{\mathbf{T}'(t)}{\left|\mathbf{T}'(t)\right|}$$

$$\mathbf{B}(t) = \mathbf{T}(t) \times \mathbf{N}(T)$$



# Example

$$\mathbf{r}(t) = \cos(t)\,\mathbf{i} + \sin(t)\,\mathbf{j} + t\,\mathbf{k}$$




$$\mathbf{r}'(t) = -\sin t\,\mathbf{i} + \cos t\,\mathbf{j} + \mathbf{k} \qquad |\mathbf{r}'(t)| = \sqrt{2}$$

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|} = \frac{1}{\sqrt{2}}(-\sin t\,\mathbf{i} + \cos t\,\mathbf{j} + \mathbf{k})$$

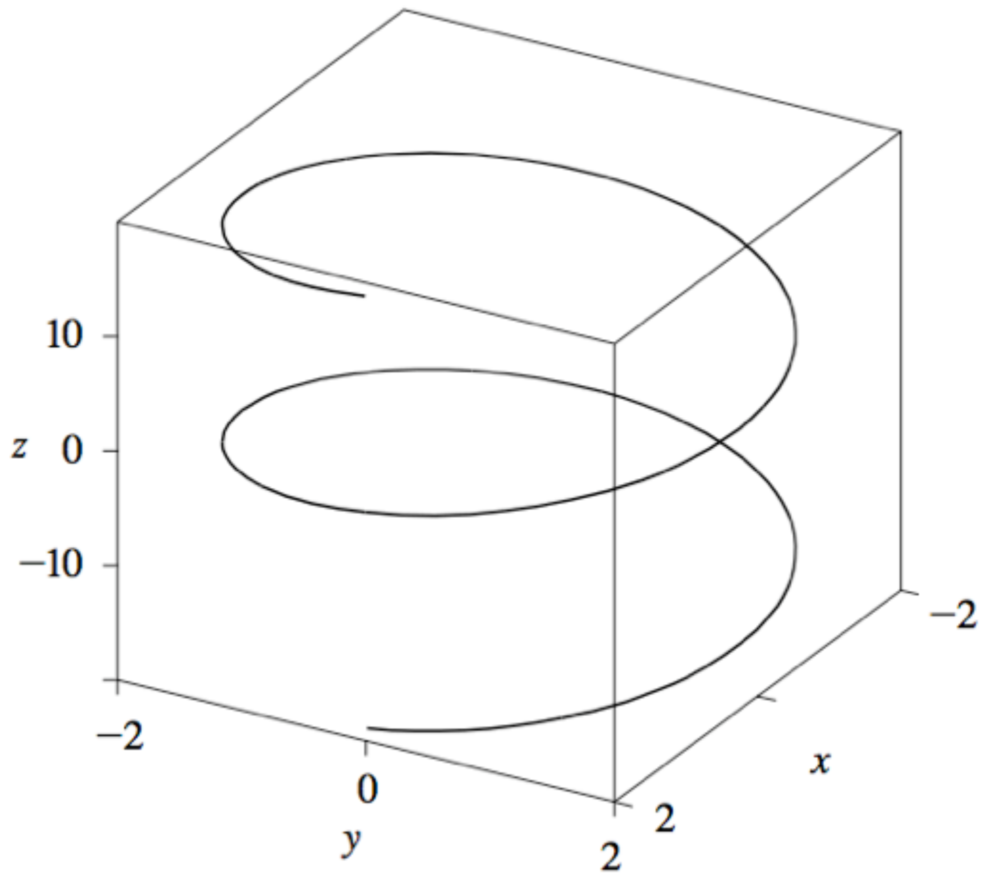
$$\mathbf{T}'(t) = \frac{1}{\sqrt{2}}(-\cos t\,\mathbf{i} - \sin t\,\mathbf{j}) \qquad |\mathbf{T}'(t)| = \frac{1}{\sqrt{2}}$$

$$\mathbf{N}(t) = \frac{\mathbf{T}'(t)}{|\mathbf{T}'(t)|} = -\cos t\,\mathbf{i} - \sin t\,\mathbf{j} = \langle -\cos t, -\sin t, 0\rangle$$

$$\mathbf{B}(t) = \mathbf{T}(t) \times \mathbf{N}(t) = \frac{1}{\sqrt{2}}\begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -\sin t & \cos t & 1 \\ -\cos t & -\sin t & 0 \end{bmatrix} = \frac{1}{\sqrt{2}}\langle \sin t, -\cos t, 1\rangle$$

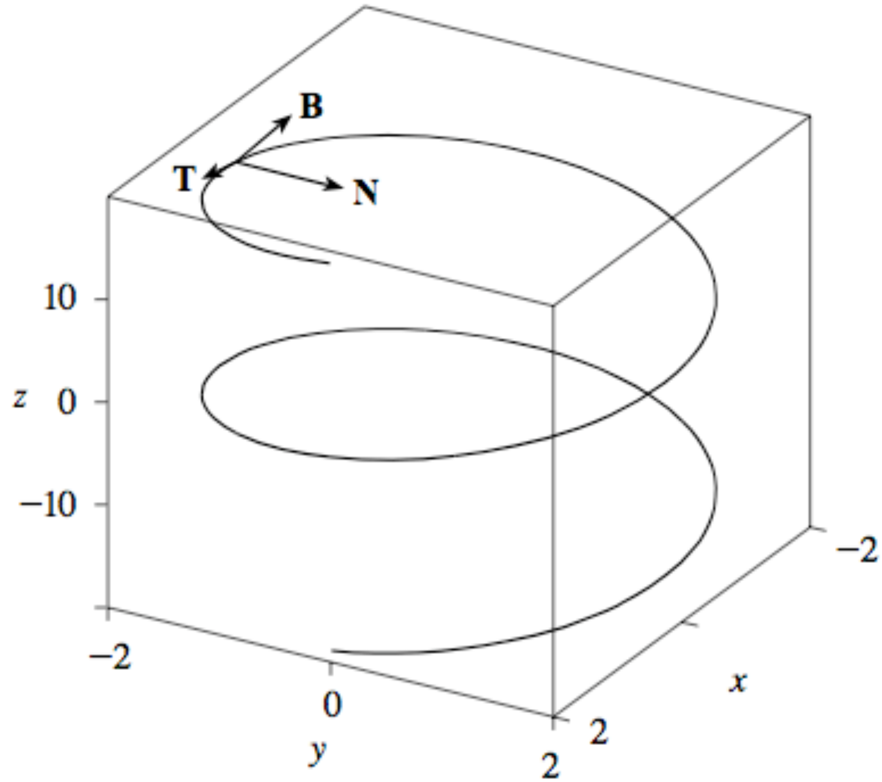
# Example

If $\mathbf{r}(t) = 2\cos t\,\mathbf{i} + 2\sin t\,\mathbf{j} + 3t\,\mathbf{k}$, find $\mathbf{T}$, $\mathbf{N}$, and $\mathbf{B}$. Sketch $\mathbf{T}$, $\mathbf{N}$, and $\mathbf{B}$ when $t = \frac{3\pi}{2}$.



# Example

$$\mathbf{T} = \left\langle \frac{2\sqrt{13}}{13}, 0, \frac{3\sqrt{13}}{13} \right\rangle, \mathbf{N} = \langle 0, -1, 0 \rangle, \mathbf{B} = \left\langle -\frac{3\sqrt{13}}{13}, 0, \frac{2\sqrt{13}}{13} \right\rangle$$

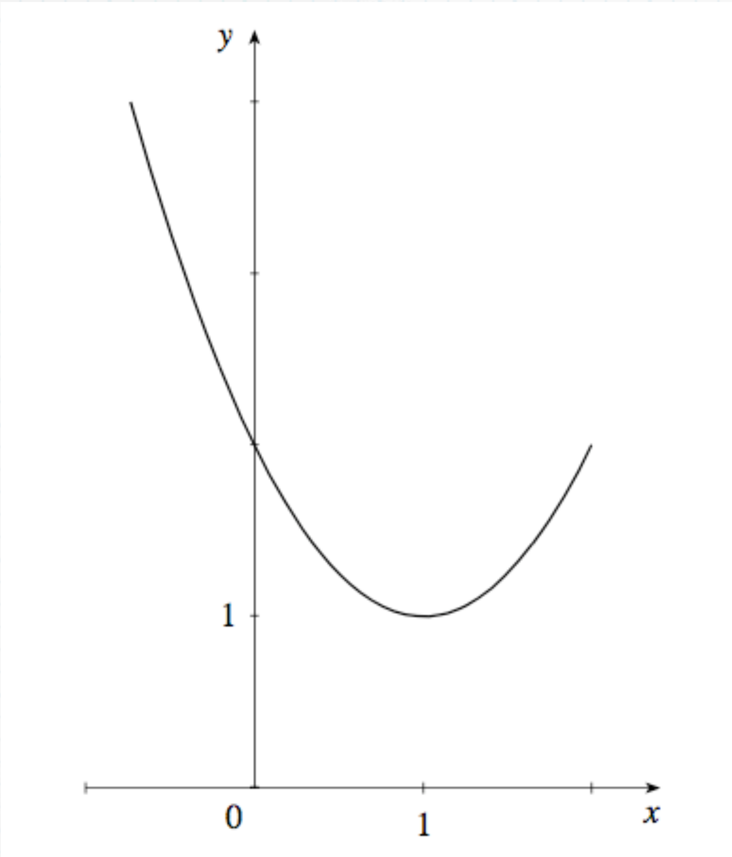


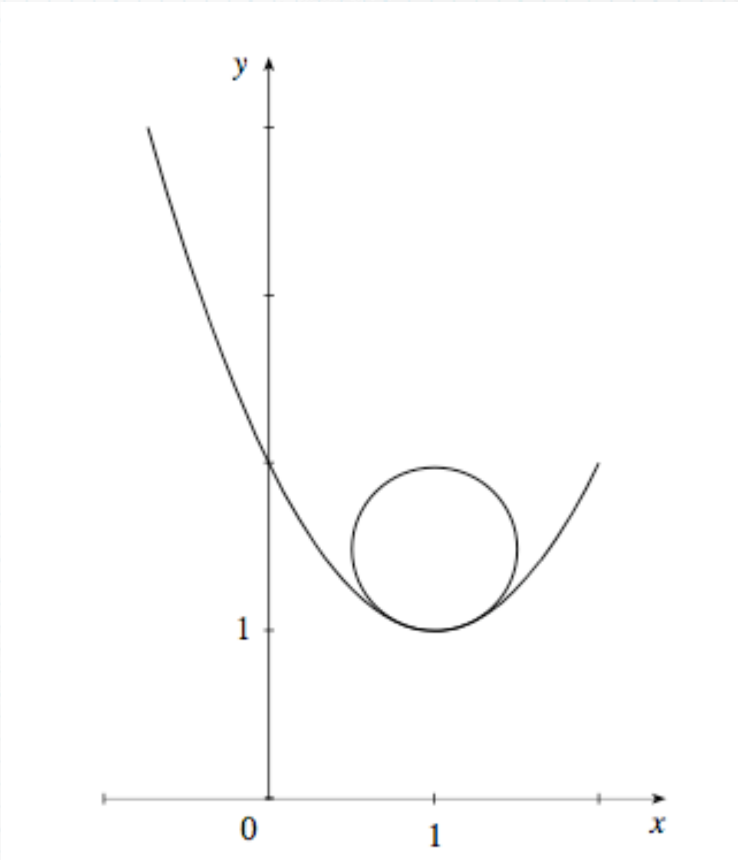
# Formulas

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{|\,\mathbf{r}'(t)\,|} \qquad \mathbf{N}(t) = \frac{\mathbf{T}'(t)}{|\,\mathbf{T}'(t)\,|} \qquad \mathbf{B}(t) = \mathbf{T}(t) \times \mathbf{N}(t)$$

$$\kappa = \left|\frac{d\mathbf{T}}{ds}\right| = \frac{|\,\mathbf{T}'(t)\,|}{|\,\mathbf{r}'(t)\,|} = \frac{|\,\mathbf{r}'(t) \times \mathbf{r}''(t)\,|}{|\,\mathbf{r}'(t)\,|^3}$$

# Osculating Circle

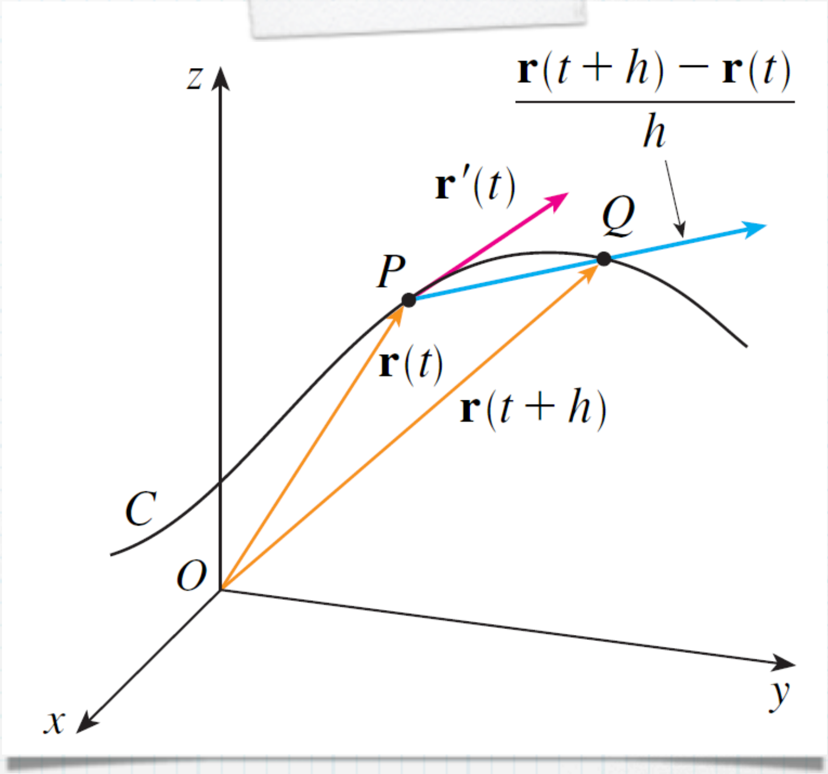


# Section 10.4

## Motion in Space

# Velocity Vector




$$\mathbf{v}(t) = \lim_{h \to 0} \frac{\mathbf{r}(t+h) - \mathbf{r}(t)}{h} = \mathbf{r}'(t)$$

# Speed

$$|\,\mathbf{v}(t)\,| = |\,\mathbf{r}'(t)\,| = \frac{ds}{dt} = \text{rate of change of distance with respect to time}$$

# Formulas

$$\mathbf{r}(t)$$

$$\mathbf{v}(t) = \mathbf{r}'(t)$$

$$\mathbf{a}(t) = \mathbf{v}'(t) = \mathbf{r}''(t)$$

$$\mathbf{r}(t) = \mathbf{r}(t_0) + \int_{t_0}^{t} \mathbf{v}(u)\,du$$

$$\mathbf{v}(t) = \mathbf{v}(t_0) + \int_{t_0}^{t} \mathbf{a}(u)\,du$$

# Projectile Motion

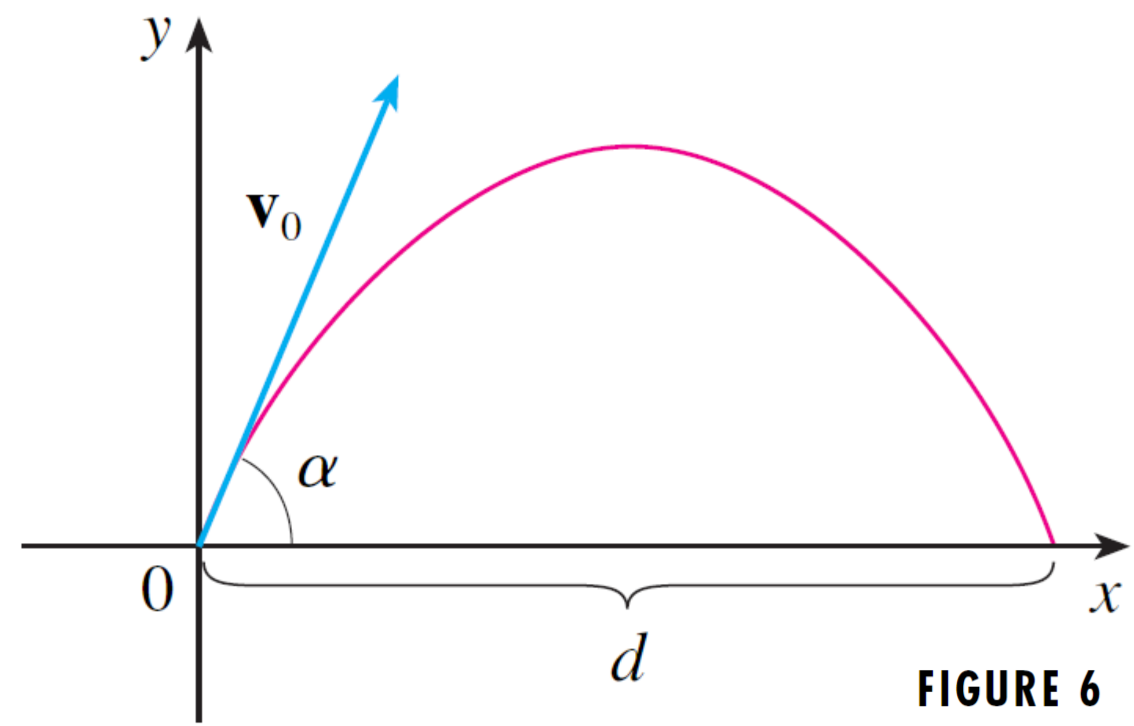
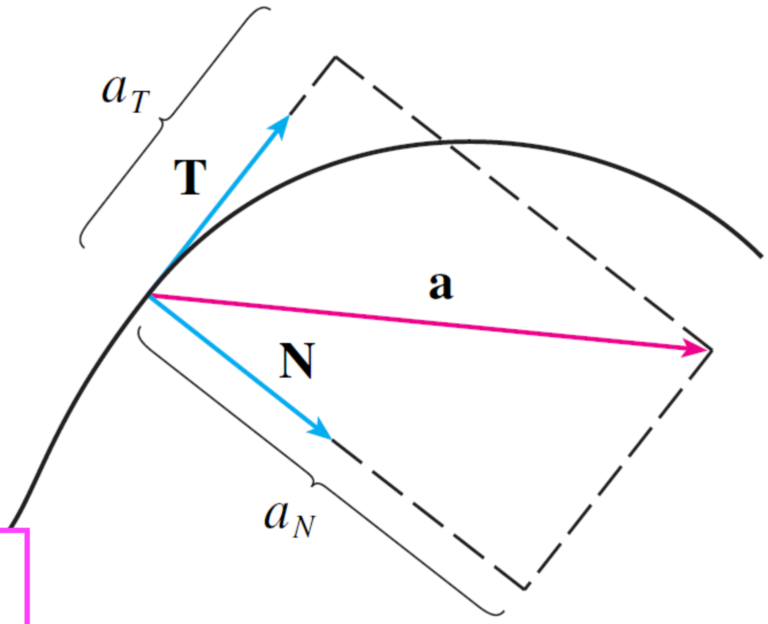



FIGURE 6

# Components of Acceleration

* When we study the motion of a particle, it is often useful to resolve the acceleration into two components,
  * one in the direction of the tangent and
  * the other in the direction of the normal.

# Components of Acceleration

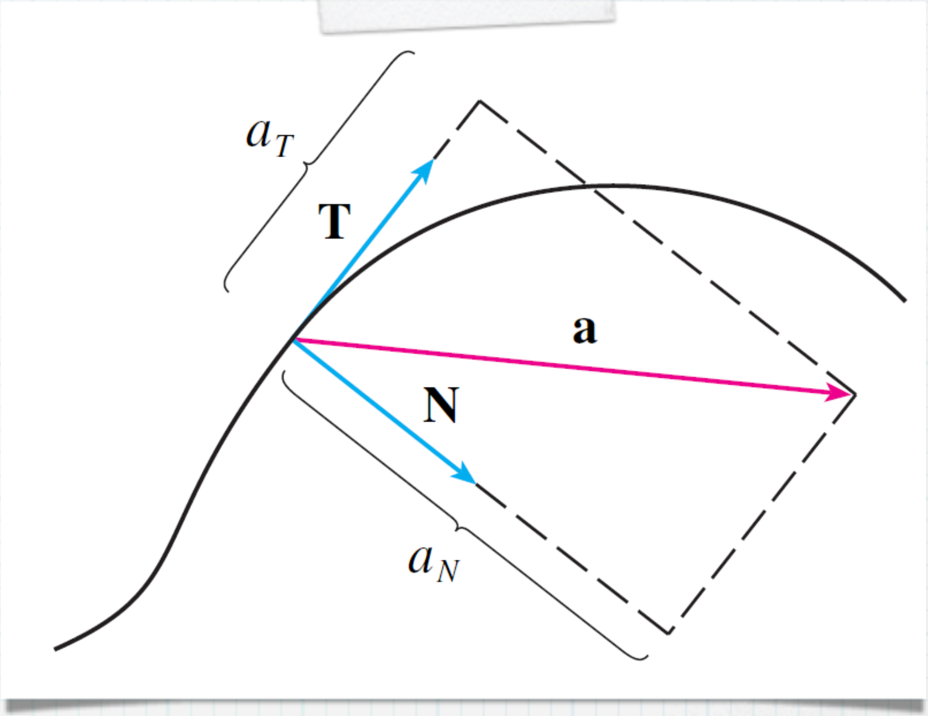



$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|} = \frac{\mathbf{v}(t)}{|\mathbf{v}(t)|} = \frac{\mathbf{v}}{v}$$

$$\kappa = \frac{|\mathbf{T}'|}{|\mathbf{r}'|} = \frac{|\mathbf{T}'|}{v} \qquad \text{so} \qquad |\mathbf{T}'| = \kappa v$$

$$\mathbf{a} = v'\mathbf{T} + \kappa v^2 \mathbf{N}$$

# Components of Acceleration




$$a_T = v' = \frac{\mathbf{v} \cdot \mathbf{a}}{v} = \frac{\mathbf{r}'(t) \cdot \mathbf{r}''(t)}{|\mathbf{r}'(t)|}$$

$$a_N = \kappa v^2 = \frac{|\mathbf{r}'(t) \times \mathbf{r}''(t)|}{|\mathbf{r}'(t)|^3} |\mathbf{r}'(t)|^2 = \frac{|\mathbf{r}'(t) \times \mathbf{r}''(t)|}{|\mathbf{r}'(t)|}$$

# Components of Acceleration

**THEOREM 1 Tangential and Normal Components of Acceleration** Let $\mathbf{a} = \mathbf{r}''(t)$ and $\mathbf{v} = \mathbf{r}'(t)$ be the acceleration and velocity vectors of a path $\mathbf{r}(t)$ at time $t$. Let $v(t) = \|\mathbf{v}(t)\|$ be the speed at time $t$. Then the coefficients $a_{\mathbf{T}} = v'(t)$ and $a_{\mathbf{N}} = \kappa(t)v(t)^2$ in the decomposition

$$\mathbf{a} = a_{\mathbf{T}}\mathbf{T} + a_{\mathbf{N}}\mathbf{N}$$

are given by the formulas

$$a_{\mathbf{T}} = \frac{\mathbf{a} \cdot \mathbf{v}}{\|\mathbf{v}\|}$$

$$a_{\mathbf{N}} = \frac{\|\mathbf{a} \times \mathbf{v}\|}{\|\mathbf{v}\|} = \sqrt{\|\mathbf{a}\|^2 - |a_{\mathbf{T}}|^2}$$

# Proof

**Proof** Recall that $\mathbf{T} = \dfrac{\mathbf{v}}{\|\mathbf{v}\|}$. Since $\mathbf{T} \cdot \mathbf{T} = 1$ and $\mathbf{N} \cdot \mathbf{T} = 0$, we obtain (4) as follows:

$$\frac{\mathbf{a} \cdot \mathbf{v}}{\|\mathbf{v}\|} = \mathbf{a} \cdot \mathbf{T} = (a_{\mathbf{T}}\mathbf{T} + a_{\mathbf{N}}\mathbf{N}) \cdot \mathbf{T} = a_{\mathbf{T}}\mathbf{T} \cdot \mathbf{T} + a_{\mathbf{N}}\mathbf{N} \cdot \mathbf{T} = a_{\mathbf{T}}$$

On the other hand, $\mathbf{T} \times \mathbf{T} = 0$, so

$$\frac{\mathbf{a} \times \mathbf{v}}{\|\mathbf{v}\|} = \mathbf{a} \times \mathbf{T} = (a_{\mathbf{T}}\mathbf{T} + a_{\mathbf{N}}\mathbf{N}) \times \mathbf{T} = a_{\mathbf{T}}\mathbf{T} \times \mathbf{T} + a_{\mathbf{N}}\mathbf{N} \times \mathbf{T} = a_{\mathbf{N}}(\mathbf{N} \times \mathbf{T})$$

Note that $\mathbf{N} \times \mathbf{T}$ is a unit vector since $\mathbf{T}$ and $\mathbf{N}$ are orthogonal (see the marginal note). Thus

$$\frac{\|\mathbf{a} \times \mathbf{v}\|}{\|\mathbf{v}\|} = a_{\mathbf{N}}\|\mathbf{N} \times \mathbf{T}\| = a_{\mathbf{N}}$$

This proves the first equality in (5). Finally, since $\mathbf{T}$ and $\mathbf{N}$ are orthogonal unit vectors, $\mathbf{a}$ is the hypotenuse of a right triangle with sides of length $a_{\mathbf{T}}$ and $a_{\mathbf{N}}$ (see Figure 4), and the Pythagorean Theorem gives

$$\|\mathbf{a}\|^2 = |a_{\mathbf{T}}|^2 + |a_{\mathbf{N}}|^2$$

This yields the second equality in (5), namely $a_{\mathbf{N}} = \sqrt{\|\mathbf{a}\|^2 - |a_{\mathbf{T}}|^2}$. ■

# Review

| | |
|---|---|
| Unit tangent vector | $\mathbf{T}(t) = \dfrac{\mathbf{v}(t)}{\lVert\mathbf{v}(t)\rVert}$ |
| Unit normal vector | $\mathbf{N}(t) = \dfrac{\mathbf{T}'(t)}{\lVert\mathbf{T}'(t)\rVert}$ |
| Decomposition of acceleration | $\mathbf{a}(t) = a_{\mathbf{T}}(t)\mathbf{T}(t) + a_{\mathbf{N}}(t)\mathbf{N}(t)$ |
| Tangential component | $a_{\mathbf{T}} = v'(t) = \dfrac{\mathbf{a}\cdot\mathbf{v}}{\lVert\mathbf{v}\rVert}$ |
| Normal component | $a_{\mathbf{N}} = \kappa(t)v(t)^2 = \dfrac{\lVert\mathbf{a}\times\mathbf{v}\rVert}{\lVert\mathbf{v}\rVert}$ |
| | $a_{\mathbf{N}}\,\mathbf{N} = \mathbf{a} - a_{\mathbf{T}}\,\mathbf{T} = \mathbf{a} - \left(\dfrac{\mathbf{a}\cdot\mathbf{v}}{\mathbf{v}\cdot\mathbf{v}}\right)\mathbf{v}$ |

# Example

* Position Vector
* Velocity Vector
* Acceleration Vector
* Speed

$$\mathbf{r}(t) = t^2\,\mathbf{i} + t^2\,\mathbf{j} + t^3\,\mathbf{k}$$

$$\mathbf{r}'(t) = 2t\,\mathbf{i} + 2t\,\mathbf{j} + 3t^2\,\mathbf{k}$$

$$\mathbf{r}''(t) = 2\,\mathbf{i} + 2\,\mathbf{j} + 6t\,\mathbf{k}$$

$$|\mathbf{r}'(t)| = \sqrt{8t^2 + 9t^4}$$

- Tangential Component of Acceleration

$$a_T = \frac{\mathbf{r}'(t) \cdot \mathbf{r}''(t)}{|\mathbf{r}'(t)|} = \frac{8t + 18t^3}{\sqrt{8t^2 + 9t^4}}$$

- Normal Component of Acceleration

$$a_N = \frac{|\mathbf{r}'(t) \times \mathbf{r}''(t)|}{|\mathbf{r}'(t)|} = \frac{6\sqrt{2}t^2}{\sqrt{8t^2 + 9t^4}}$$

$$\mathbf{r}'(t) \times \mathbf{r}''(t) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2t & 2t & 3t^2 \\ 2 & 2 & 6t \end{vmatrix} = 6t^2\,\mathbf{i} - 6t^2\,\mathbf{j}$$

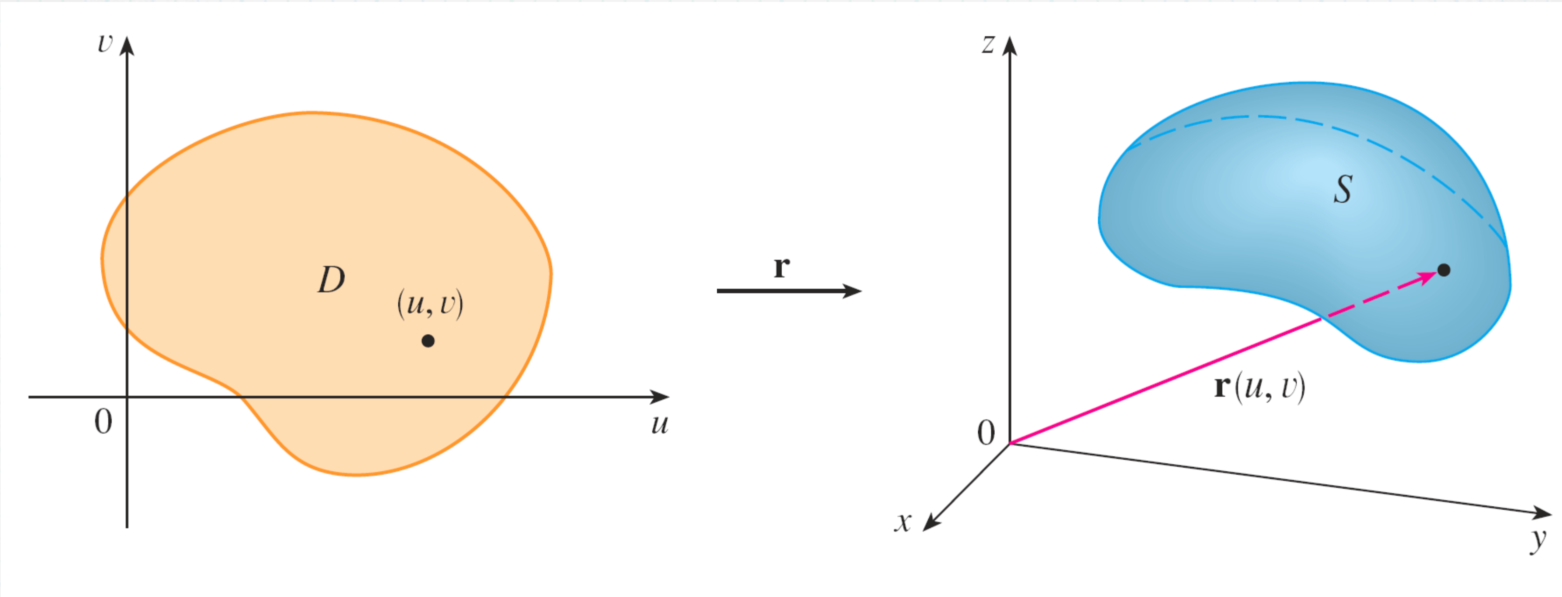
# Section 10.5

## Parametric Surfaces

# Parametric Surfaces

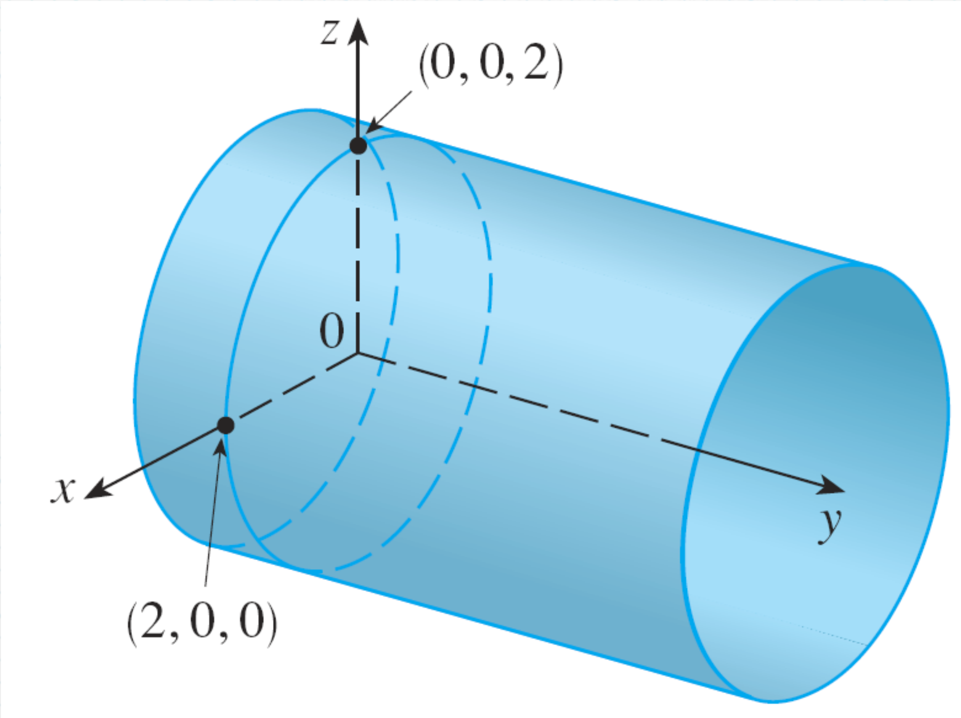
* We suppose that r(u, v) =< x(u, v), y(u, v),z(u, v)> is a vector-valued function defined on a region D in the uv-plane.

* As (u, v) varies throughout D, r(u, v) traces out a parametric surface S.



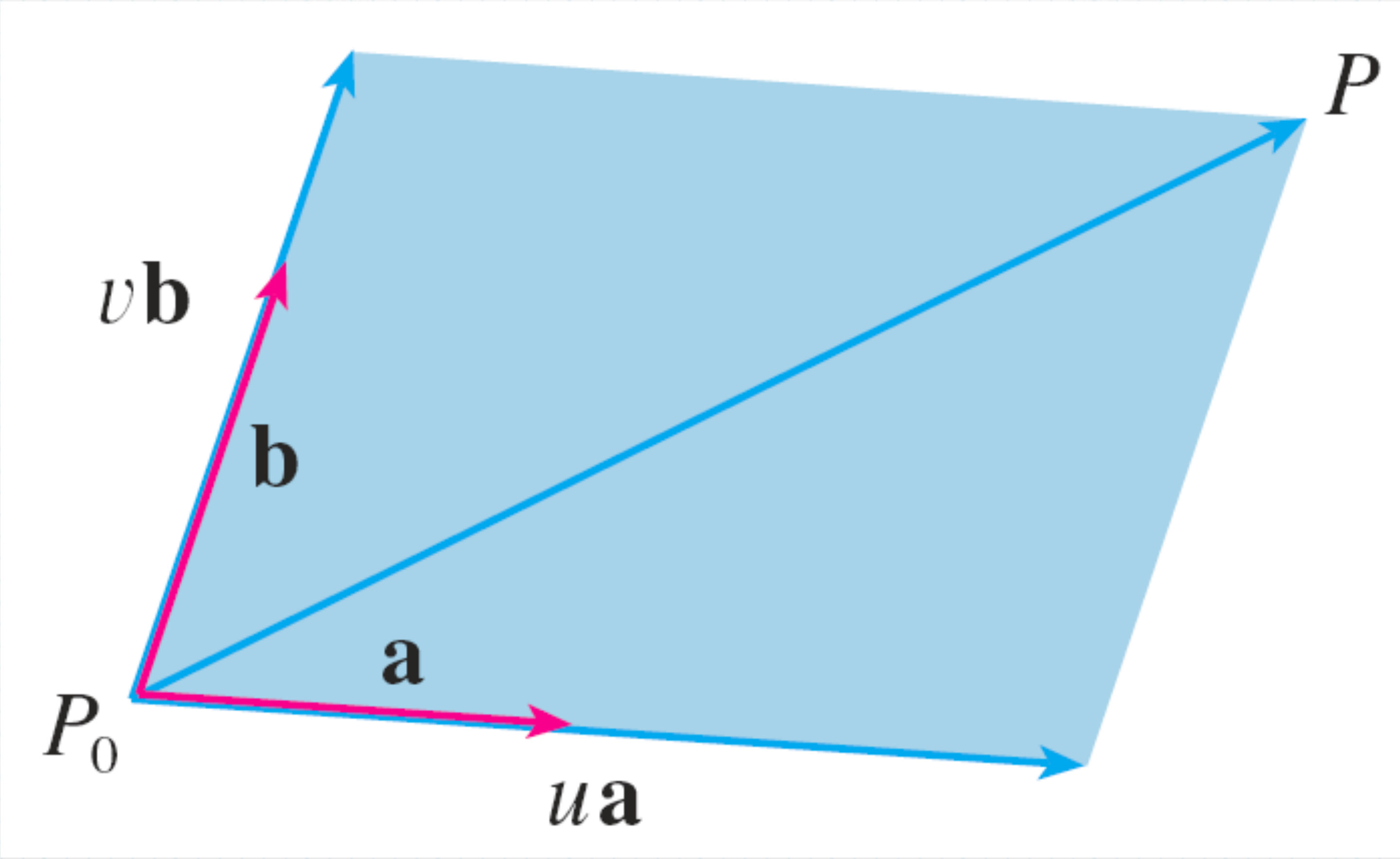
# Example: Cylinder

r(u, v) = <2cos(u), v, 2sin(u)>



# Example: Plane

$r(u,v)=r_0+ua+vb$

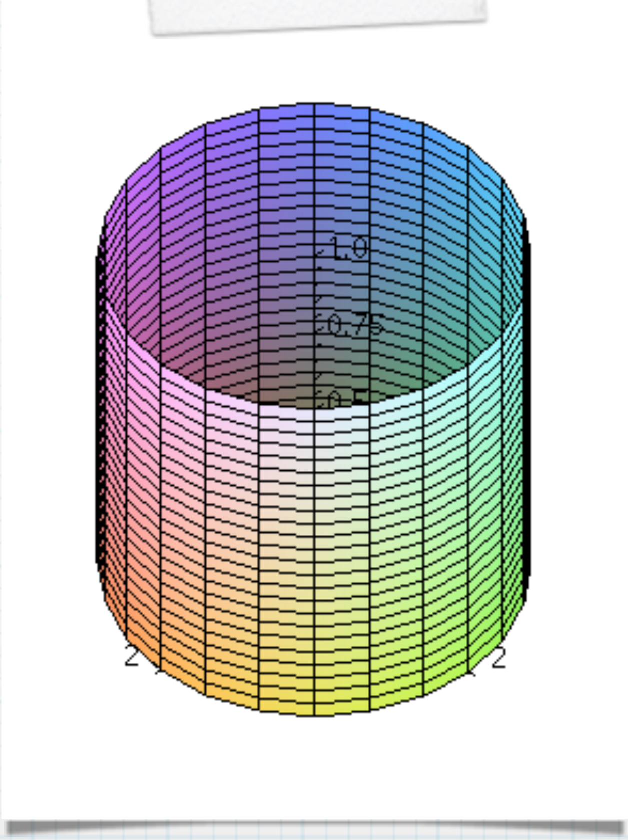


# Example: Another Cylinder

$x^2 + y^2 = 4 \quad 0 \leq z \leq 1$

$x=2\cos(\theta)$
$y=2\sin(\theta)$
$z=z$

$r(\theta,z)=\langle 2\cos(\theta), 2\sin(\theta), z\rangle$



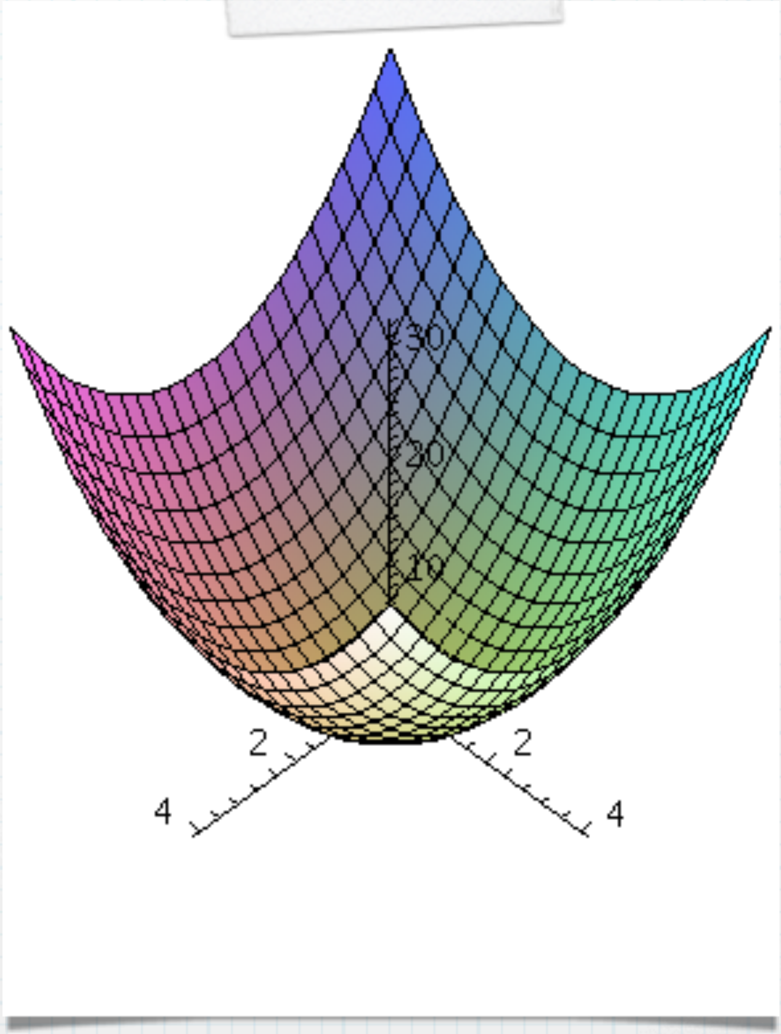

Maple command
plot3d ( <2cos(θ), 2sin(θ), z>, θ=0..2π,z=0..1 ,axes=normal)

# Example: Elliptic Paraboloid

$z = x^2 + 2y^2$

$x=x$
$y=y$
$z = x^2 + 2y^2$

$r(x,y)= \langle x, y, x^2 + 2y^2 \rangle$




**Maple command**
plot3d ( < x, y, x^2 + 2y^2>, x=-4..4,y=-4..4,axes=normal)