Name-
Math 213, May 2020, Test 2

This is an open book exam. You may consult your notes / class notes. You may use CoCalc / GeoGebra software as calculational aids. But you may not consult other people.

For the purposes of partial credit, show plenty of calculations. Add extra blank pages if you need to.

1. A metal plate is situated in the xy plane and occupies the rectangle $0 \leq x \leq 10,\ 0 \leq y \leq 8$ where x and y are measured in centimeters. The temperature at the point (x, y) in the plate is $T(x, y)$, where T is measured in degrees Celsius. Temperatures at equally spaced points on the plate were measured and recorded. A portion of those measurements is shown in the table below.

	y=2	y=4	y=6
x=4	60 C	64	62
x=6	84	76	72
x=8	100	90	82

a. Estimate the values of the partial derivatives $T_x(6,4)$ and $T_y(6,4)$. Give *units* with your answers.

b. Using these partial derivative estimates, write down the linear approximation of $T(x, y)$ for points near $(6, 4)$.

c. Using your linear approximation, calculate the value of $T(7, 5)$.

a.) $T_x \approx \dfrac{\Delta T}{\Delta x} = \dfrac{90-64}{8-4} = 6.5\ {}^\circ C/cm$

$T_y \approx \dfrac{\Delta T}{\Delta y} = \dfrac{72-84}{6-2} = -3.0\ {}^\circ C/cm$

b.) $\Delta T = T_x \Delta x + T_y \Delta y$

$\underline{T(x,y)} - T(6,4) = T_x (x-6) + T_y (y-4)$

$T(x,y) = \underset{\uparrow}{76} + 6.5(x-6) - 3(y-4)$

$T(6,4)$

c.) $T(7,5) = 76 + 6.5(7-6) - 3(5-4)$

$= 79.5\ {}^\circ C$

2. You are told that a certain function, $f(x,y)$ has these partial derivatives:

$$\frac{\partial f}{\partial x} = 3y^2 + 2x + \ln(x * y) + 1 \tag{1}$$

$$\frac{\partial f}{\partial y} = 6xy + \frac{x}{y} \tag{2}$$

Carry out Clairaut's test to decide if $f(x,y)$ is a continuous function or not. Show the results of your test, and state your conclusion about whether it is continuous or not.

See below

3. Consider the function

$$f(x,y) = \frac{x+y}{|x|+|y|} \tag{3}$$

(a) Evaluate the function at the following locations:

i. $f(1,1)$

ii. $f(1,-1)$

iii. $f(-1,1)$

iv. $f(-1,-1)$

(b) Either find $\lim_{(x,y)\to(0,0)} f(x,y)$ or show that it doesn't exist.

i. $\frac{1+1}{1+1} = \frac{2}{2} = 1$

iii $\frac{-1+1}{1+1} = \frac{0}{2} = 0$

ii $\frac{1-1}{1+1} = \frac{0}{2} = 0$

iv $\frac{-1-1}{1+1} = \frac{-2}{2} = -1$

b.) consider $x=y=t$, $t \to 0$ from positive values

$f(x,y) = \frac{t+t}{2|t|} = \frac{2t}{2t} = 1$

but if $t<0$

$f(x,y) = \frac{t+t}{2|t|} = \frac{2t}{-2t} = -1$

limit does not exist!

2.)

Clairaut sez, A continuous function must obey: $\dfrac{\partial}{\partial y}\dfrac{\partial f}{\partial x} = \dfrac{\partial}{\partial x}\dfrac{\partial f}{\partial y}$

$$\frac{\partial}{\partial y}\frac{\partial f}{\partial x} = \frac{\partial}{\partial y}\left[3y^2 + 2x + \ln(x \cdot y) + 1\right]$$

$$= 6y + 0 + \frac{1}{xy} \cdot x + 0$$

$$= 6y + \frac{1}{y} \leftarrow$$

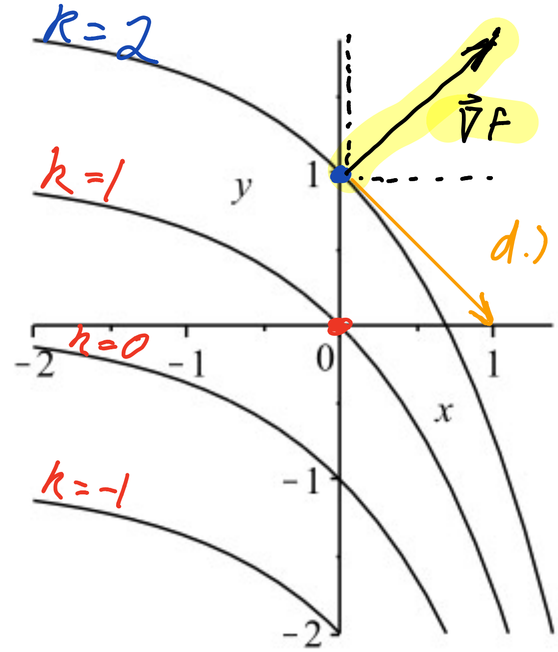
Since

$f_{xy} = f_{yx}$

$$\frac{\partial}{\partial x}\frac{\partial f}{\partial y} = \frac{\partial}{\partial x}\left(6xy + \frac{x}{y}\right) \Rightarrow$$ f is continuous

$$= 6y + \frac{1}{y} \leftarrow$$

4. Several level curves for the function $f(x, y) = e^x + y$ are shown below.



a. What is the value of $k = f(x, y)$ on each of the level curves shown? Label each level curve with the value of k *on* that level curve.

b. Find a formula, $y = f(x)$, for the level curve shown that passes through the point $(0, 1)$.

c. At the point $(0, 1)$ draw and label the vector $\vec{\nabla} f(0, 1)$. Be accurate as to the direction and length of this vector.

d. On the graph, draw and label a vector that starts at $(0, 1)$ and points in a direction in which $f(x, y)$ remains constant. What are the components of the vector you found?

e. Find the directional derivative of $f(x, y)$ at the point (0,1) in the direction of $\vec{\mathbf{u}} = 3\hat{\imath} + 4\hat{\jmath}$.

f. What is the maximum possible value of any directional derivative at $(0, 1)$?

a.) One of the curves passes through the origin: $(x,y) = (0,0)$. So, for that one,
$k = f(0,0) = e^0 + 0 = 1 + 0 = 1$ so $k = 1$

b.) Same approach as above to find k:
The curve passes through $(0,1)$ so
$k = f(0,1) = e^0 + 1 = 1 + 1 = 2$
Therefore, we have $2 = e^x + y$. Solving for y:
$$y = 2 - e^x$$

c.) $\vec{\nabla} f = \langle f_x, f_y \rangle = \langle e^x, 1 \rangle$
where $f_x = \dfrac{\partial f}{\partial x} = \dfrac{\partial}{\partial x}(e^x + y) = e^x$
$f_y = \dfrac{\partial}{\partial y}(e^x + y) = 1$

$$\vec{\nabla} f \Big|^{(x,y)=(0,1)} = \langle e^0, 1 \rangle = \langle 1, 1 \rangle$$

d.) Any vector which is perpendicular to $\vec{\nabla} f$ will be tangent to a contour line.

for example:

$$\langle 1, -1 \rangle \cdot \langle 1, 1 \rangle = 0$$

Or also, $\langle -1, 1 \rangle$; $\langle \frac{1}{2}, -\frac{1}{2} \rangle$; ....

any vector $\langle a, b \rangle$ where $a = -b$

e.) $D_{\hat{u}} f = \vec{\nabla} f \cdot \hat{u}$

if $\vec{u} = \langle 3, 4 \rangle$ the $|\vec{u}| = \sqrt{3^2 + 4^2} = 5$

so $\hat{u} = \dfrac{\vec{u}}{u} = \dfrac{\langle 3, 4 \rangle}{5} = \langle \frac{3}{5}, \frac{4}{5} \rangle = \langle 0.6, 0.8 \rangle$

$$D_{\hat{u}} f = \vec{\nabla} f \cdot \hat{u} = \langle 1, 1 \rangle \cdot \langle \tfrac{3}{5}, \tfrac{4}{5} \rangle = \frac{3}{5} + \frac{4}{5} = \frac{7}{5} = 1.4$$

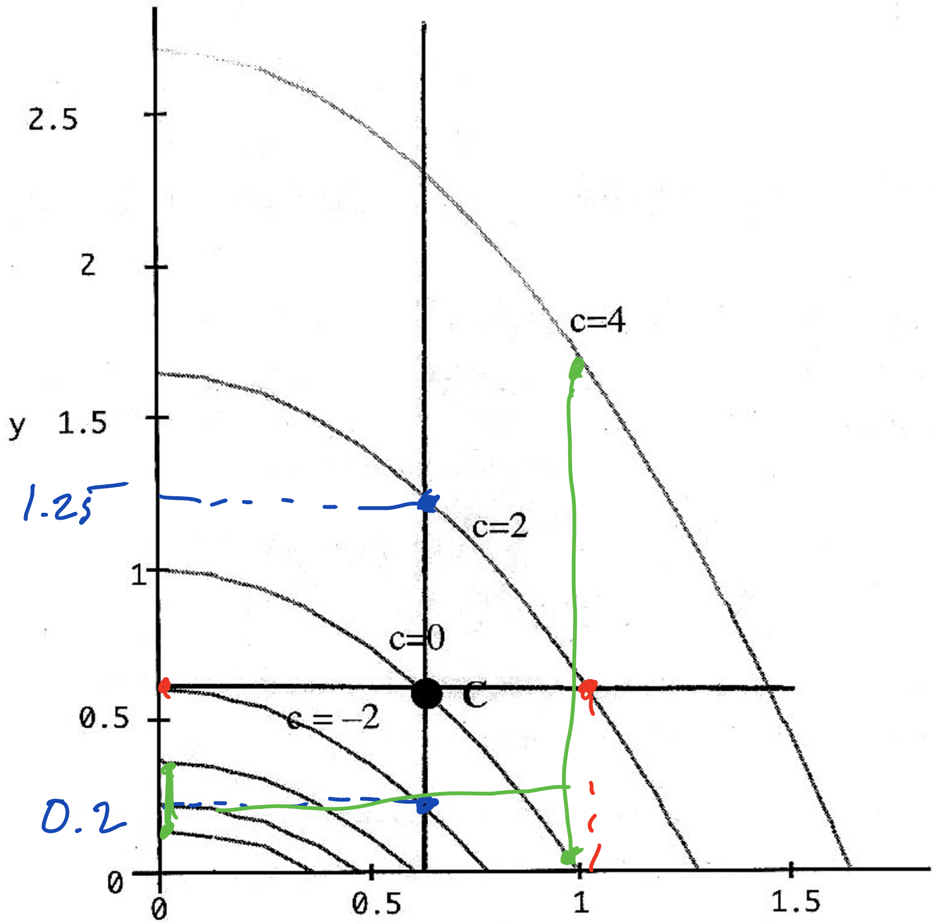
f.) $|\vec{\nabla} f|$ is the greatest possible slope

$$|\vec{\nabla} f| = |\langle 1, 1 \rangle| = \sqrt{1^2 + 1^2} = \sqrt{2} = 1.414$$

good, this should be greater than (or equal to) any Directional derivative in another direction, such as part e.)

$1.414 > 1.4$

5. The following is a contour graph of a function $z = f(x, y)$. Values $z = c$ are indicated on the plot.



Estimate from the contour plot the values of these partial derivatives at point C.

$f_x = ?$ $\frac{\Delta f}{\Delta x} = \frac{2-(-2)}{1-0} \simeq +4$

$f_y = ?$ $\frac{\Delta f}{\Delta y} \simeq \frac{2-(-2)}{1.25-0.2} = +3.8$

Determine whether the following derivatives are positive or negative or 0 at the point C. *Explain* your answers.

f_{xx} concave up > 0

f_{yy} concave up > 0

f_{xy} f_y steep $\Rightarrow$ less steep as x increases $\Rightarrow f_{xy} < 0$

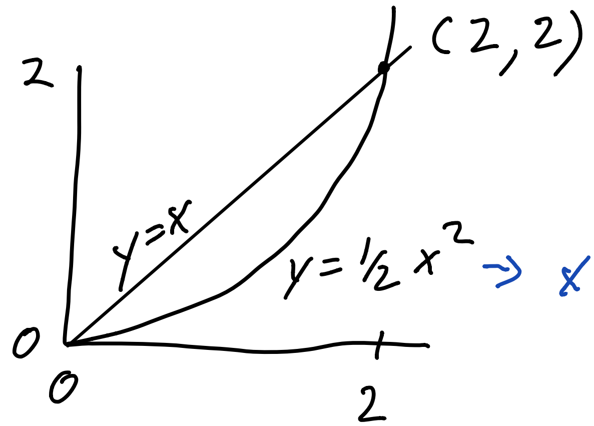
6. $\int_0^2 \int_{y=\frac{1}{2}x^2}^{x} \left(x+3y^2\right)\, \cancel{dy\,dx}$. $dx\,dy$

a. Sketch the region of integration in the xy plane, labeling all important points, and functions you graph.

b. Evaluate the integral by hand, showing all steps in the computation, including antiderivatives. Your answer should be a single fraction or decimal number (to the nearest thousand'th).

c. Write the integral that reverses the order of integration. (You **do not need to evaluate it**.)

a.)



$\rightarrow x(y)=\sqrt{2y}$

c.) $\int_0^2 \int_{x=y}^{\sqrt{2y}} (x+3y^2)\, dy\,dx$

7. In the figure, the contour plot of $f(x,y) = x^2 + xy + y^2$ is shown. (Dark blue are regions where f is low and in orange-colored regions, f is high.) Aso, a path (in red) given by $g(x,y) = y - x^2 = -3$ is drawn in.

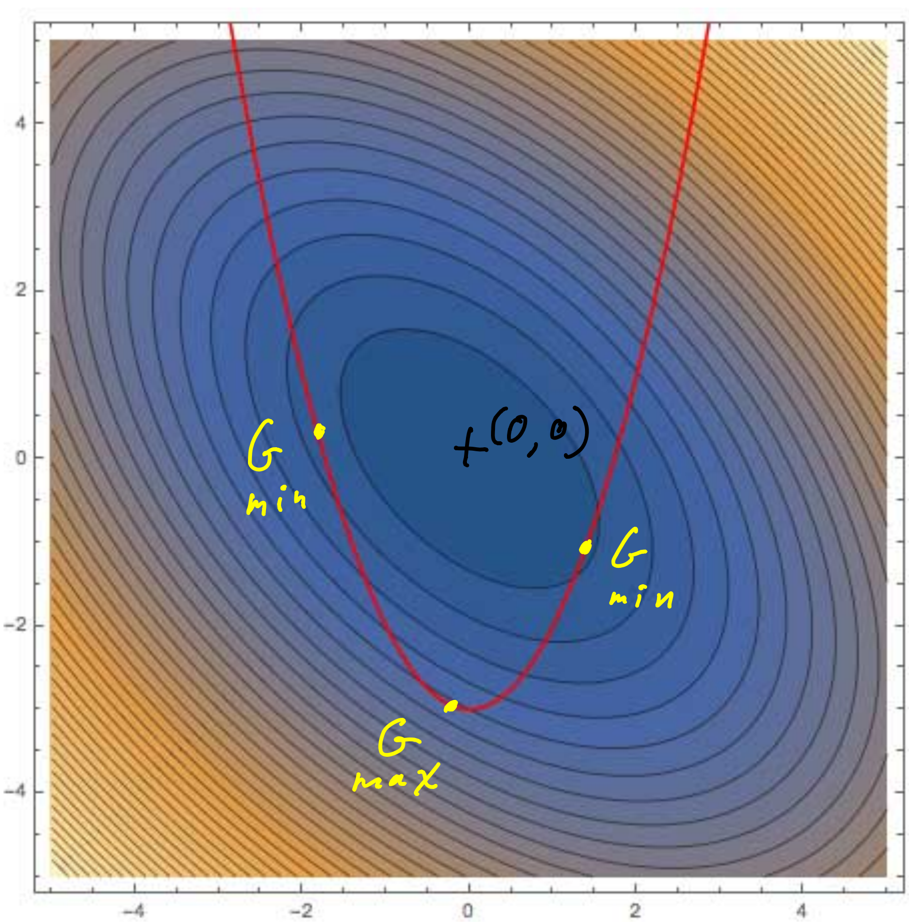
   a. Find the coordinates of all the critical points of the function $f(x,y)$. For each one, state whether it is a maximum, a minimum, or neither. If a maximum or minimum, state whether it is a global or local one.

   b. The red line is a path through the "landscape" of the surface, f, going up and down, according to the contour plot. Estimate the position(s) of any maxima or minima in surface height along the red path. Mark each one on the path on the diagram with a point and a "G" for "Guess", and write whether it appears to be a maximum or a minimum.

   c. Calculate, and show the gradients of $f(x,y)$ and $g(x,y)$.

   d. Now, using the method of Lagrange multipliers, look for maxima or minima along the level curve $g(x,y) = -3$ (there may be more than one): Write out the 3 equations that such points must satisfy.

   e. Solve the equations for x, y, and λ for each such point. You may use software (e.g. CoCalc) to solve these equations, or you may solve them by hand. Write down the coordinates of the point(s) on the level curve which are solutions (to the nearest hundredth), and mark each one on the diagram with a letter "L".



a.) $f_x = 2x + y = 0 \Rightarrow y = -2x$

$f_y = x + 2y = 0 \quad x + 2y = x - 4x = 0 \Rightarrow x = 0$ global

only one solution, $(x,y) = (0,0)$, a minimum b.) Look for points where the path $\approx$ parallel to a nearby contour line. 3 such places.

c.) $f = x^2 + xy + y^2$

$\vec{\nabla} f = \langle 2x+y, x+2y \rangle$

$g = y - x^2$

$\vec{\nabla} g = \langle -2x, 1 \rangle$

d.) $\vec{\nabla} f = \lambda \vec{\nabla} g$ $\quad (\star)$

$2x + y = -2\lambda x$ $\quad$ (x-components of $\star$)

$x + 2y = \lambda$ $\quad$ (y-components of $\star$

$y - x^2 = -3$ $\quad$ $(g(x,y) = -3)$