Vectors [9.2]

Many physical quantities--displacement, velocity, force to name a few--have both a magnitude and a **direction**.

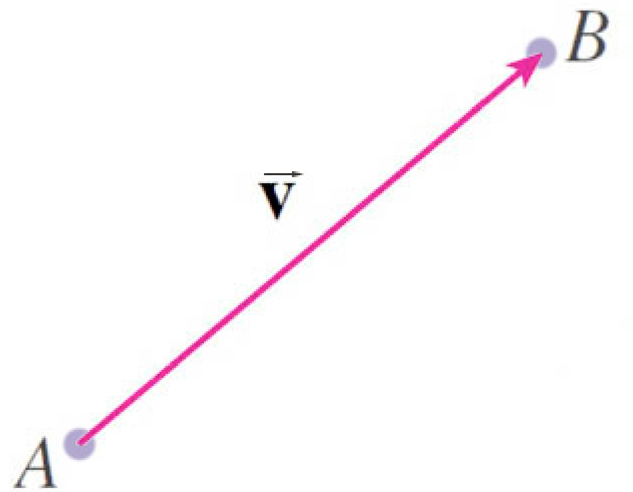
- vector terminology (2-d)
- vectors and scalars
- scalar multiplication of a vector
- vector addition and subtraction
- vector components
- using vectors to define lines

Vectors

A **vector** is a quantity that has both a magnitude and a direction.

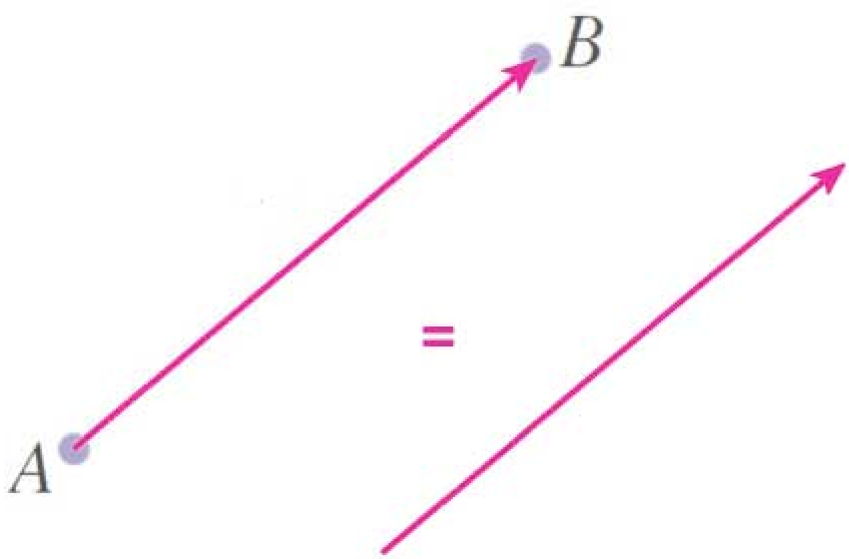
A vector is visually represented by an arrow or a directed line segment which connects two points: the **tail** (initial point / base) and the **head** (terminal point / tip).

- The **length** of the arrow represents the **magnitude** of the vector (same as the distance between tail and head) and
- the arrow **points** in the **direction** of the vector.



Examples: wind velocity $\vec{\mathbf{v}}$, force $\vec{\mathbf{F}}$, magnetic field $\vec{\mathbf{B}}$, acceleration $\vec{\mathbf{a}}$.

Two vectors are **equal** if they have the **same direction** and the **same magnitude**. (Their positions don't matter.)



Scalar

A **scalar number** is, for our purposes, usually the same as a **real number**.

A scalar number can be used to represent a quantity that does *not* have a direction.

For example: time t, temperature T, the probability that a fish in the ocean is a shark, p_{shark}.

Notation

Scalar quantities are typically typeset in italics:

$$t, T, p_{shark}$$

Vector quantities are typeset either in bold:

$$\mathbf{u}, \mathbf{v}, \mathbf{F}$$

and/or with an arrow on the top:

$$\vec{\mathbf{u}}, \vec{\mathbf{v}}, \vec{\mathbf{F}}$$

> **I encourage you** to write an arrow over a letter to distinguish a vector from a scalar, when you're writing mathematics out by hand.

In physics

- "Speed", v, is a scalar. It can have a value like 30 mph or 6 ft / sec and is always a positive number.
- "Velocity", $\vec{\mathbf{v}}$, is a vector quantity. The magnitude of velocity, $|\vec{\mathbf{v}}| = v$, is speed. The velocity might have a magnitude like 4 meters / sec *and* a direction, such as north-east.

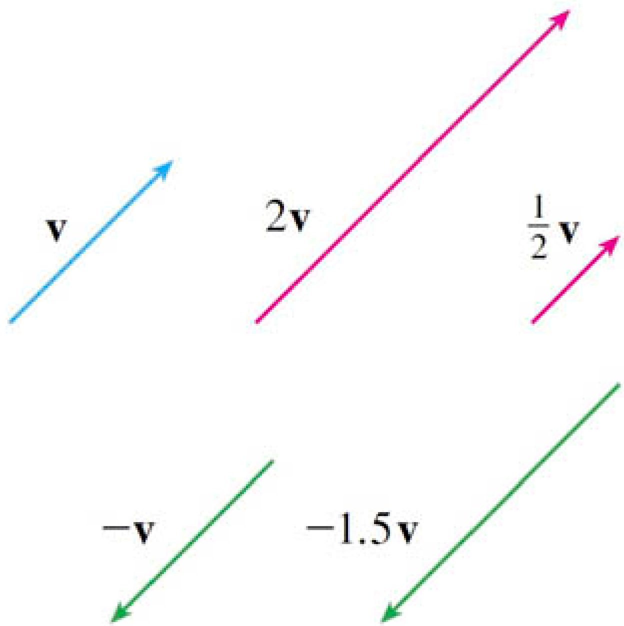
There is a general convention of using the scalar letter to indicate the magnitude (length) of the corresponding vector quantity. For example, acceleration, $\vec{\mathbf{a}}$ has a direction. And its magnitude is a:

$$a = |\vec{\mathbf{a}}| = \text{the "norm" of } \vec{\mathbf{a}}. \tag{1}$$

Some interchangeable terms for the "length" of a vector:

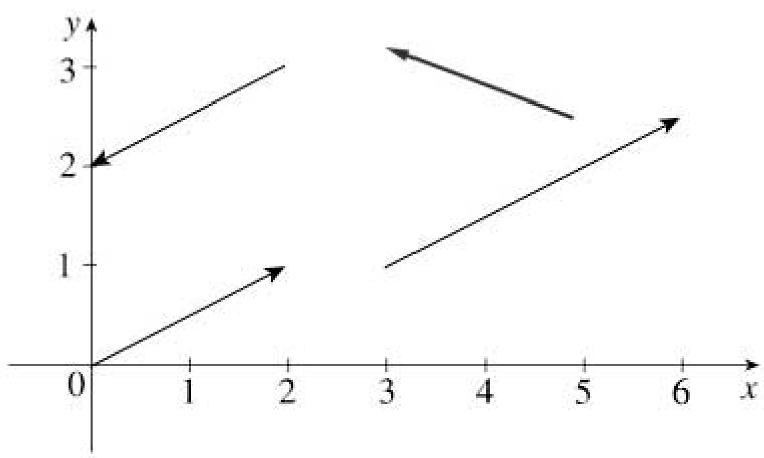
- magnitude
- norm
- measure

Scalar multiplication



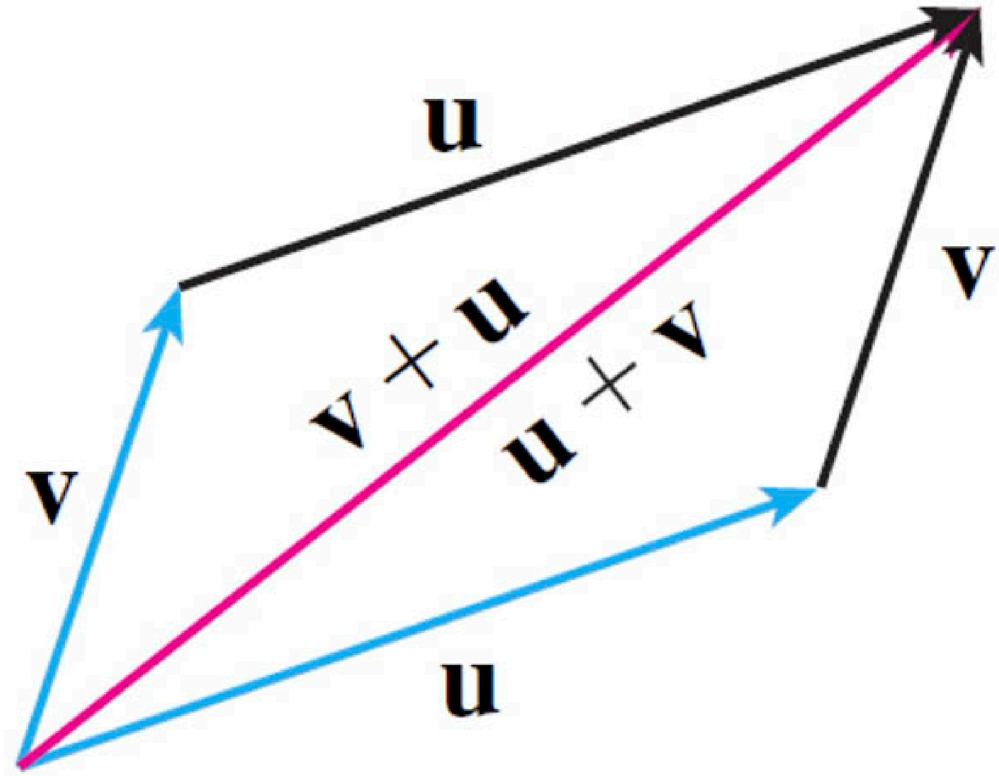
If c is a scalar, and $\vec{\mathbf{v}}$ is a vector, then the **scalar multiple** $c\vec{\mathbf{v}}$ is the vector with:

- a length of $|c|$ times the length of $\vec{\mathbf{v}}$, and
- a direction which is:
  - the **same** as $\vec{\mathbf{v}}$ if $c > 0$, or
  - **opposite** $\vec{\mathbf{v}}$ if $c < 0$, or



*What relationship (involving scalar multiplication) must exist between two vectors $\vec{\mathbf{a}}$ and $\vec{\mathbf{b}}$ iff [if and only if] they are **parallel**?*

Adding vectors

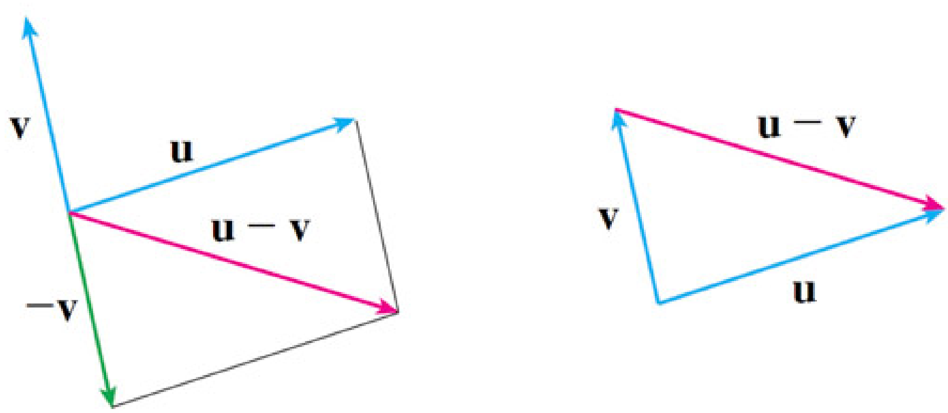


1. If $\vec{\mathbf{u}}$ and $\vec{\mathbf{v}}$ are vectors,
2. positioned such that the initial point of $\vec{\mathbf{v}}$ is at the terminal point of $\vec{\mathbf{u}}$,
3. then the *sum* $\vec{\mathbf{u}} + \vec{\mathbf{v}}$ is the vector from the initial point of $\vec{\mathbf{u}}$ to the terminal point of $\vec{\mathbf{v}}$.

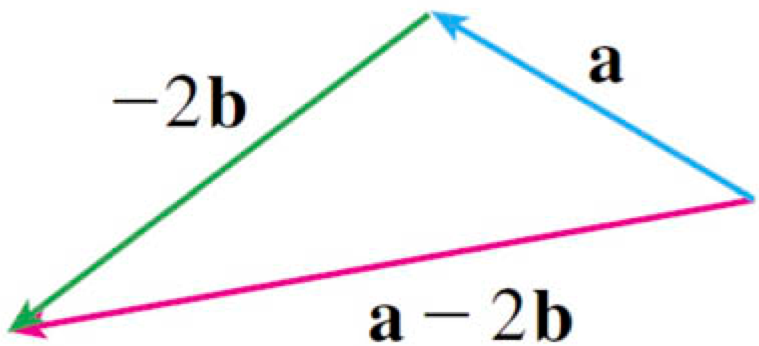
Vector subtraction

The idea is...

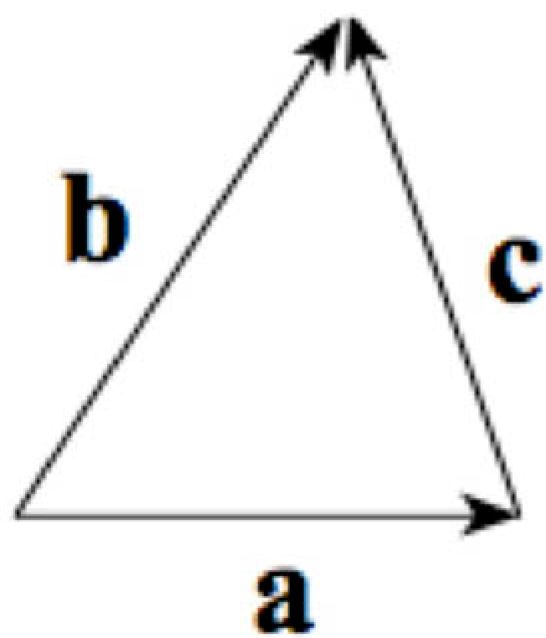
$$\vec{\mathbf{u}} - \vec{\mathbf{v}} \equiv \vec{\mathbf{u}} + (-1)\vec{\mathbf{v}}$$



Sketch $\vec{\mathbf{a}} - 2\vec{\mathbf{b}}$



Use vector addition/subtraction to come up with an expression for $\vec{\mathbf{c}}$ in terms of $\vec{\mathbf{a}}$ and $\vec{\mathbf{b}}$.



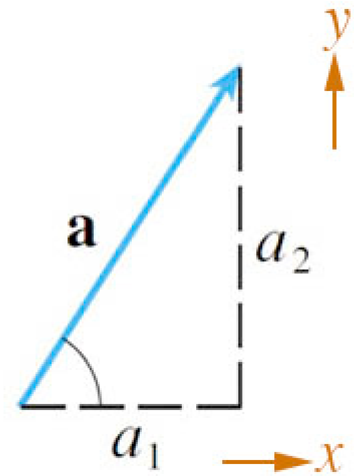
Components

The vector, $\vec{\mathbf{a}}$, is drawn in a 2-d Cartesian coordinate system. The difference between the x-coordinate of its tip and of its base is $\Delta x = a_1$. The difference between the y-coordinate of its tip and of its base is a_2. We write:

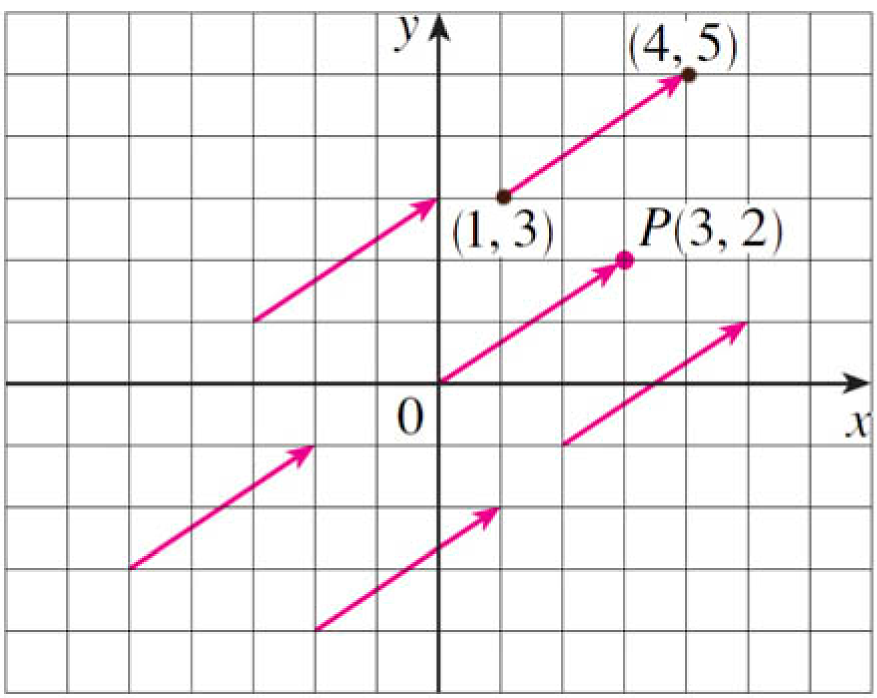
$$\vec{\mathbf{a}} = \langle a_1, a_2 \rangle \tag{2}$$

where

- a_1 is the "x-component" of the vector $\vec{\mathbf{a}}$,
- a_2 is the " y-component" of $\vec{\mathbf{a}}$.



All of these vectors have the same **components**: $\langle 3, 2\rangle$.



Are all of these vectors equal to each other, or not?

Vector components in 3-d:

$$\vec{\mathbf{a}} = \langle a_1, a_2, a_3\rangle$$

Length

...of a vector in terms of its components.

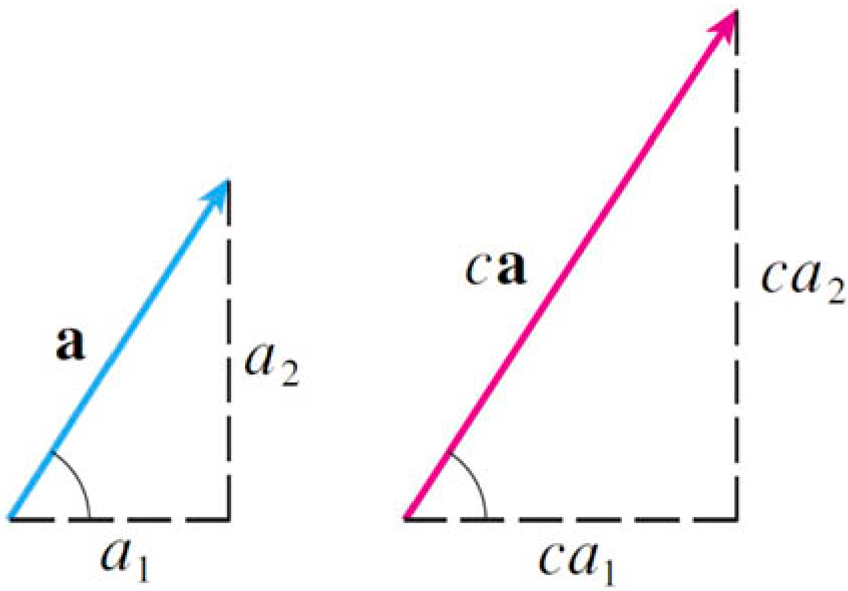
Length $|\vec{\mathbf{a}}|$ (a scalar number) of the 2-d vector $\vec{\mathbf{a}} = \langle a_1, a_2\rangle$:

$$|\vec{\mathbf{a}}| = \sqrt{a_1^2 + a_2^2}. \tag{3}$$

Length of the 3-d vector $\vec{\mathbf{a}} = \langle a_1, a_2, a_3 \rangle$:

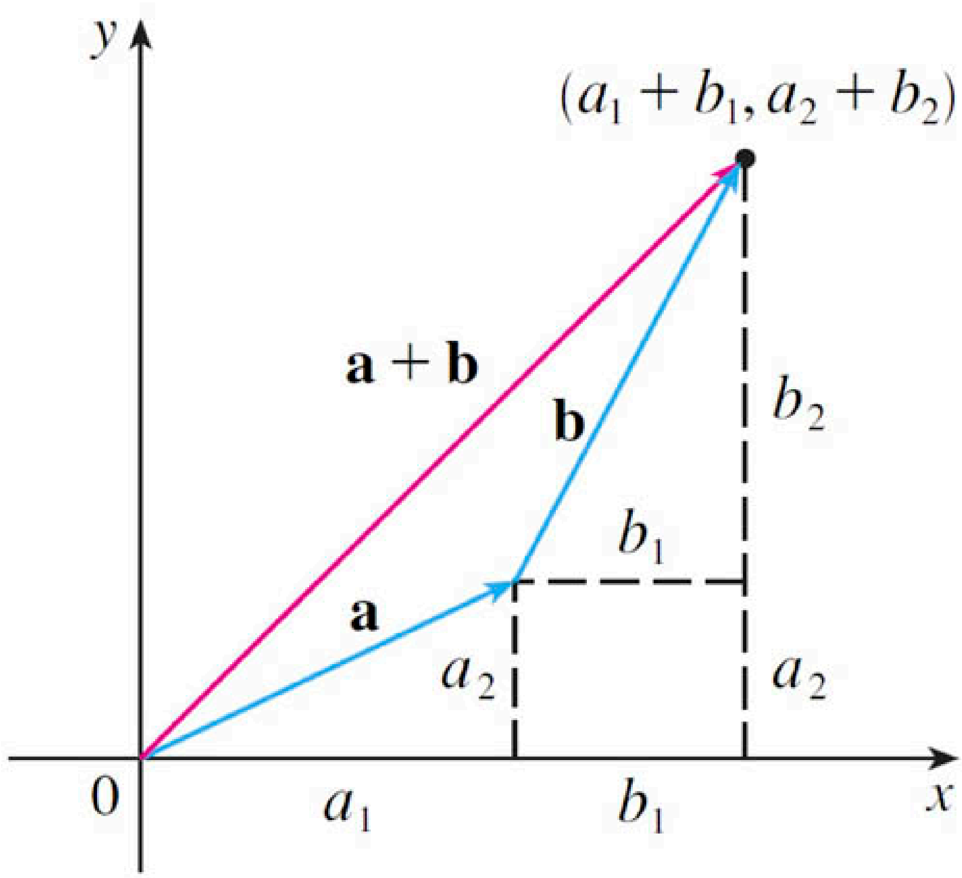
$$|\vec{\mathbf{a}}| = \sqrt{a_1^2 + a_2^2 + a_3^2}. \tag{4}$$

Scalar multiplication (components)



$$c\vec{\mathbf{a}} = \langle ca_1, ca_2 \rangle. \tag{5}$$

Vector addition (and subtraction)



$$\vec{\mathbf{a}} + \vec{\mathbf{b}} = \langle a_1 + b_1, a_2 + b_2 \rangle \tag{6}$$

Subtraction:

$$\vec{\mathbf{a}} - \vec{\mathbf{b}} = \langle a_1 - b_1, a_2 - b_2 \rangle \tag{7}$$

All these results generalize to 3-d.

More properties

tors If $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ are vectors in V_n and c and d are scalars, then

$+ \mathbf{a}$ **2.** $\mathbf{a} + (\mathbf{b} + \mathbf{c}) = (\mathbf{a} + \mathbf{b}) + \mathbf{c}$

4. $\mathbf{a} + (-\mathbf{a}) = \mathbf{0}$

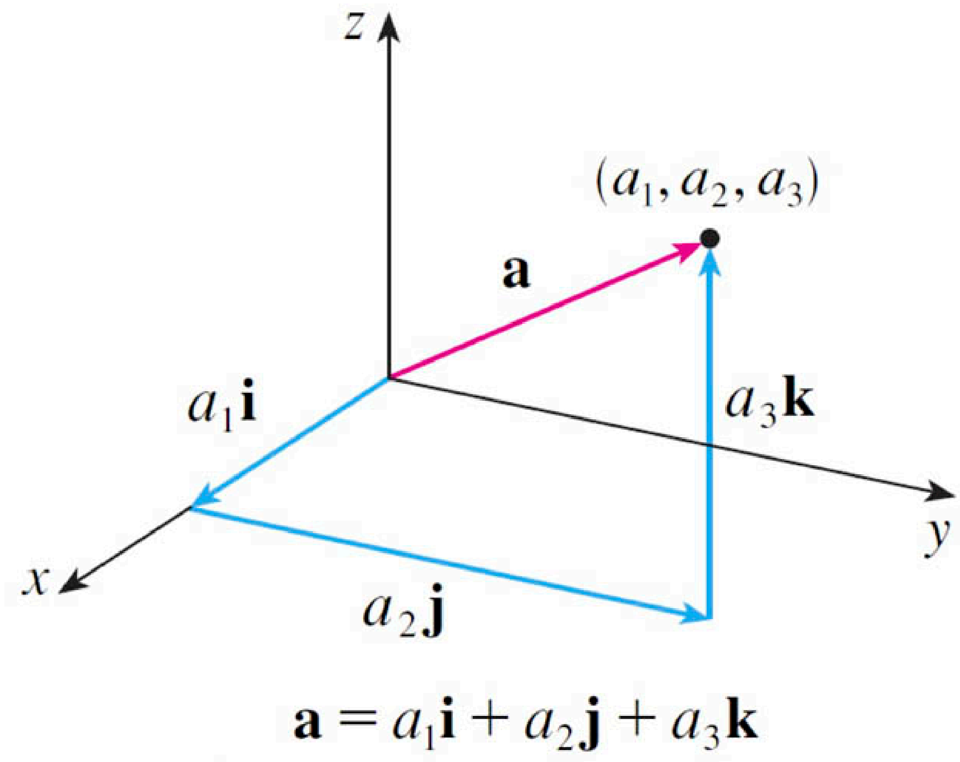
$= c\mathbf{a} + c\mathbf{b}$ **6.** $(c + d)\mathbf{a} = c\mathbf{a} + d\mathbf{a}$

$d\mathbf{a})$ **8.** $1\mathbf{a} = \mathbf{a}$

Standard basis vectors

The 3 standard "basis vectors" in a Cartesian coordinate system are:

$$\vec{\mathbf{i}} \equiv \langle 1,0,0\rangle; \quad \vec{\mathbf{j}} \equiv \langle 0,1,0\rangle; \quad \vec{\mathbf{k}} \equiv \langle 0,0,1\rangle \tag{8}$$



Notice that each of these vectors has a **length of 1**. That is, they are **unit vectors**.

A convention that I like, is to indicate unit vectors with the caret symbol up top (instead of an arrow):

$$\hat{\mathbf{i}} \equiv \langle 1,0,0\rangle \equiv \hat{\mathbf{x}}; \quad \hat{\mathbf{j}} \equiv \langle 0,1,0\rangle \equiv \hat{\mathbf{y}}; \quad \hat{\mathbf{k}} \equiv \langle 0,0,1\rangle \equiv \hat{\mathbf{z}} \tag{9}$$

Find the unit vectors

Find the unit vector...

- that points in the direction $\langle 8,0,0\rangle$.
- that points in the direction $\langle 5,5,0\rangle$.
- that is *opposite* to the direction $\langle 1,-1,3\rangle$.

In general...

If $\vec{\mathbf{a}}$ is a vector, then a unit vector, pointing in the same direction as $\vec{\mathbf{a}}$, but with a length of 1 unit is:

$$\hat{\mathbf{a}} = \frac{\vec{\mathbf{a}}}{a}. \tag{10}$$

For example, if $\vec{\mathbf{a}} = \langle 5, 5, 0 \rangle$, then $a = \sqrt{5^2 + 5^2 + 0^2} = \sqrt{50}$, so

$$\hat{\mathbf{a}} = \frac{\vec{\mathbf{a}}}{a} == \frac{\langle 5, 5, 0 \rangle}{\sqrt{50}} = \langle \frac{5}{\sqrt{50}}, \frac{5}{\sqrt{50}}, 0 \rangle. \tag{11}$$

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