Dot product [9.3]

No, no, not that kind of dot product...

There are two ways to form a **product of vectors**:

- The "dot product" of two vectors, $\vec{\mathbf{a}} \cdot \vec{\mathbf{b}}$ is a *scalar*.
- The "cross product" of two vectors, $\vec{\mathbf{a}} \times \vec{\mathbf{b}}$ is a *vector*.

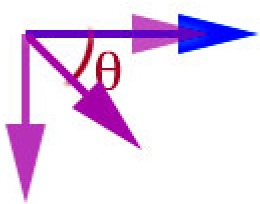
How much force goes into moving a cart?

Consider pushing a cart, with the same force, F, but in different directions:

- in the same direction as its motion.
- at right angles to its direction of motion.
- at some angle θ, relative to its direction of motion.

Brian Evans

How much does each force contribute to making the cart move?



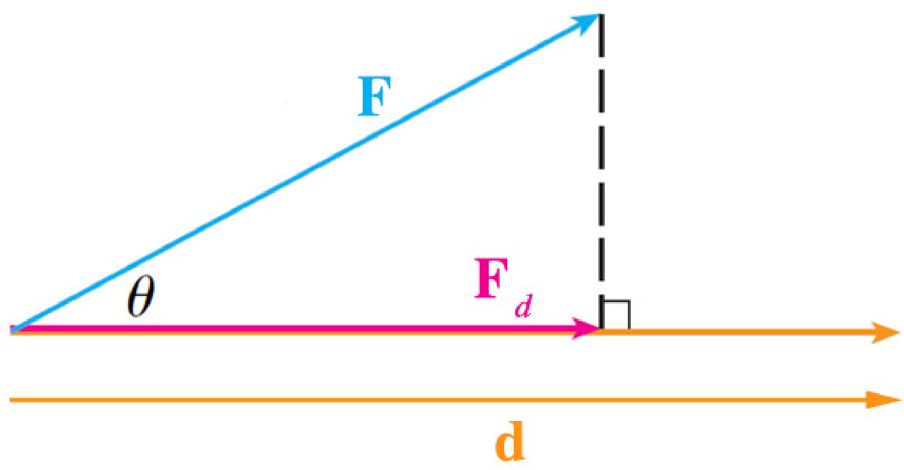
How about:

$$F \cos \theta ?? \tag{1}$$

How much energy?

"Work" in physics is the energy expended when a constant force $\vec{\mathbf{F}}$ acts to displace an object by a distance $\vec{\mathbf{d}}$.

Using the convention that $d \equiv |\vec{\mathbf{d}}|$, $F \equiv |\vec{\mathbf{F}}|$, etc...



Work is the product of the displacement and the component of the force along the direction of the displacement.

$$\text{Work} = F_d d = (F\cos\theta)d$$

We shall see shortly that this can be written as

$$\text{Work} = \vec{\mathbf{F}} \cdot \vec{\mathbf{d}}. \tag{2}$$

[Units: Newton-meter = Joule]

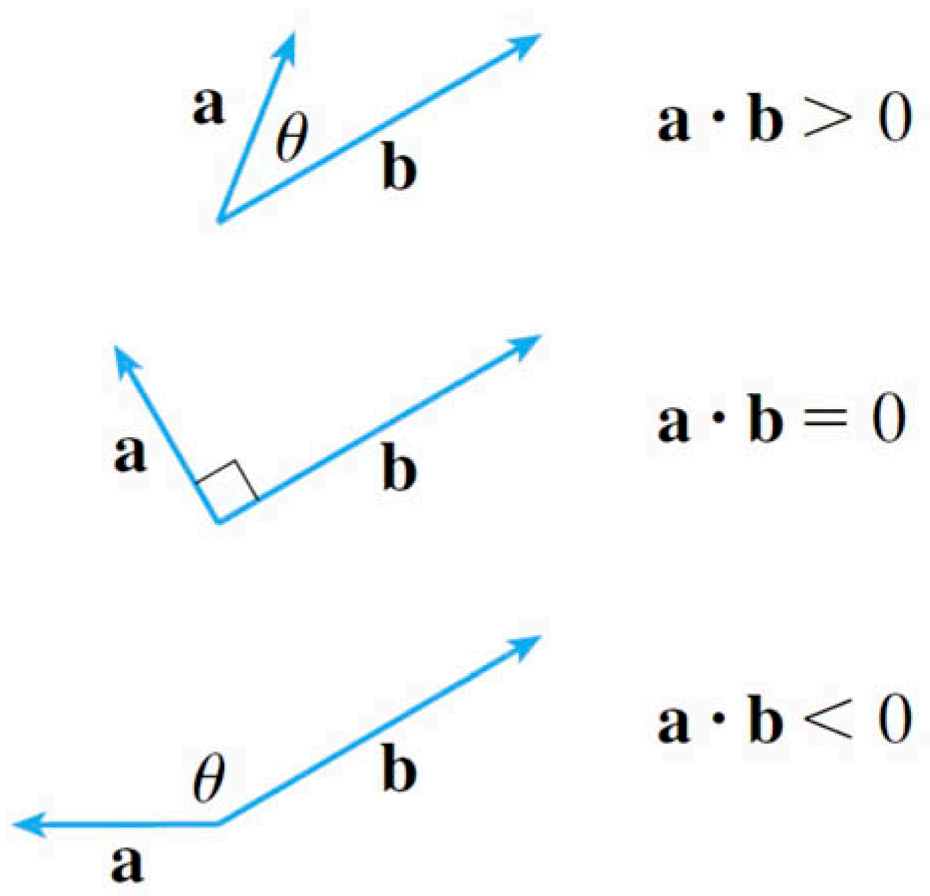
So... If the force is at right angles to the displacement no work is done? Can you think of an example?

The dot product

...of two non-zero vectors $\vec{\mathbf{a}}$ and $\vec{\mathbf{b}}$ is defined as

$$\vec{\mathbf{a}} \cdot \vec{\mathbf{b}} \equiv |\vec{\mathbf{a}}||\vec{\mathbf{b}}|\cos\theta \tag{3}$$

where θ is the **smallest possible angle** between the vectors when they're drawn with the same initial point. That is, $0 \leq \theta \leq \pi$.



Two vectors are **orthogonal** (which is a general way of saying "perpendicular") if and only if their dot product is zero.

$$\vec{\mathbf{a}} \cdot \vec{\mathbf{b}} = 0 \quad \Leftrightarrow \quad \vec{\mathbf{a}} \perp \vec{\mathbf{b}}. \tag{4}$$

Component form

The dot product of $\vec{\mathbf{a}} = \langle a_1, a_2, a_3 \rangle$ and $\vec{\mathbf{b}} = \langle b_1, b_2, b_3 \rangle$ can be written in terms of their components:

$$\vec{\mathbf{a}} \cdot \vec{\mathbf{b}} = a_1 b_1 + a_2 b_2 + a_3 b_3. \tag{5}$$

Examples:

$$\langle 2, 4 \rangle \cdot \langle 3, -1 \rangle = 2(3) + 4(-1) = 2$$

$$\langle -1, 7, 4 \rangle \cdot \langle 6, 2, \frac{1}{2} \rangle = -1(6) + 7(2) + 4(\frac{1}{2}) = 10$$

$$(\vec{\mathbf{i}} + 2\vec{\mathbf{j}} - 3\vec{\mathbf{k}}) \cdot (2\vec{\mathbf{j}} - \vec{\mathbf{k}}) = 1(0) + 2(2) - 3(-1) = 7$$

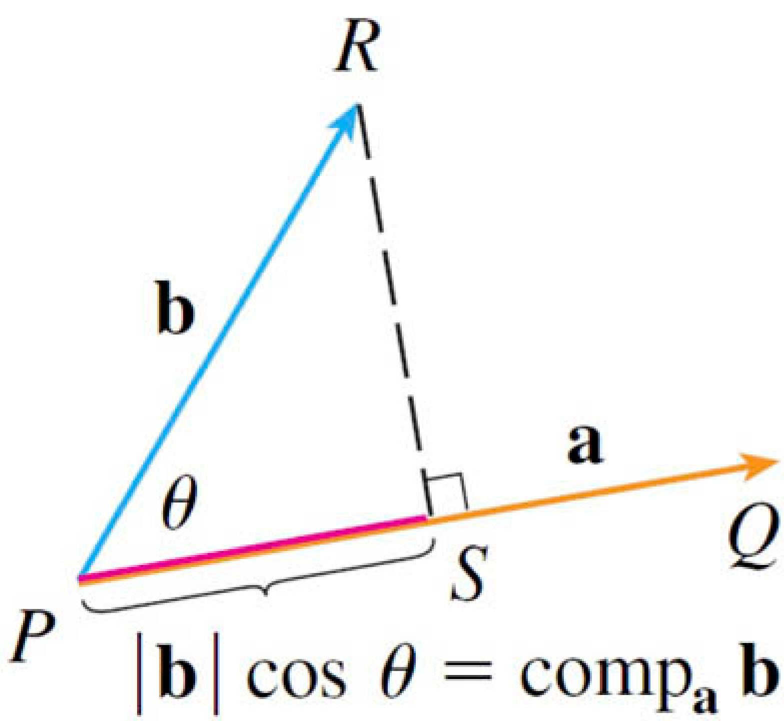
Properties of the dot product

Dot Product If **a**, **b**, and **c** are vectors in V_3 and c is a scalar, then

$|^2$ **2.** $\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a}$

$= \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \cdot \mathbf{c}$ **4.** $(c\mathbf{a}) \cdot \mathbf{b} = c(\mathbf{a} \cdot \mathbf{b}) = \mathbf{a} \cdot (c\mathbf{b})$

Component

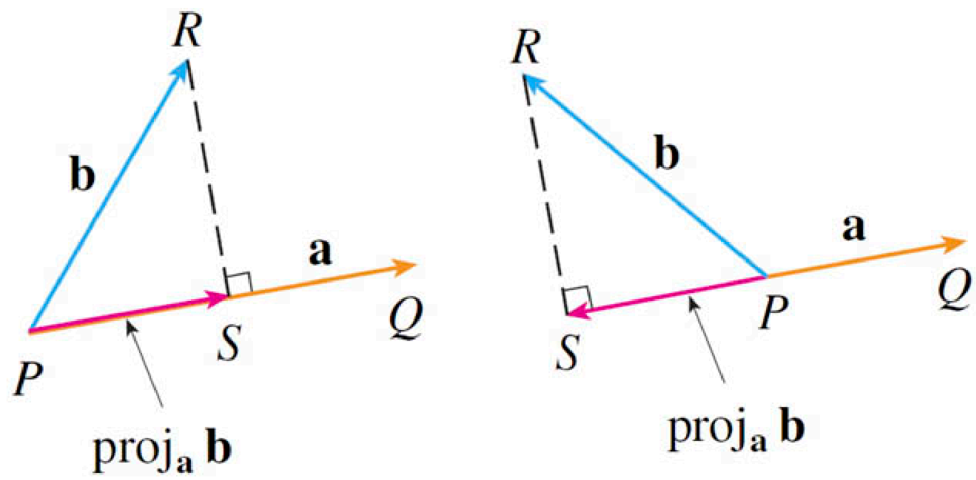


The **component** of $\vec{\mathbf{b}}$ along $\vec{\mathbf{a}}$ is a scalar:

$$\mathrm{comp}_{\vec{\mathbf{a}}}\vec{\mathbf{b}} = b\cos\theta = \frac{|\vec{\mathbf{a}}|}{|\vec{\mathbf{a}}|}|\vec{\mathbf{b}}|\cos\theta = \frac{\vec{\mathbf{a}}\cdot\vec{\mathbf{b}}}{|\vec{\mathbf{a}}|}. \tag{6}$$

...also called the *scalar projection*.

Projection



The **vector projection** of $\vec{\mathbf{b}}$ along $\vec{\mathbf{a}}$ is a **vector** which points along $\vec{\mathbf{a}}$ and has a length of $b\cos\theta = \mathrm{comp}_{\vec{\mathbf{a}}}\vec{\mathbf{b}}$.

We can construct a vector, $\hat{\mathbf{a}}$ of unit length pointing in the $\vec{\mathbf{a}}$ direction like this:

$$\hat{\mathbf{a}} = \frac{\vec{\mathbf{a}}}{|\vec{\mathbf{a}}|} \tag{7}$$

[I will frequently write $\hat{\mathbf{i}}$, $\hat{\mathbf{j}}$, $\hat{\mathbf{k}}$ for the rectangular coordinate unit vectors.]

The vector projection is:

$$\mathrm{proj}_{\vec{\mathbf{a}}}\vec{\mathbf{b}} = b\cos\theta\,\hat{\mathbf{a}} = \frac{\vec{\mathbf{a}}\cdot\vec{\mathbf{b}}}{|\vec{\mathbf{a}}|^2}\vec{\mathbf{a}}. \tag{8}$$

To do

- *The Regular Hexagon*

Image credits

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